The Time Value of Money

Dated cash flows, present value, and financial decisions

Summer 2026

Fatih Kansoy
Computational Finance & FinTech

Day 1 | Week 1

One ticket, two choices

Loto-Québec, *Gagnant à vie*, 2025.

In 2025, **Brenda Aubin-Vega** won the top lottery prize and had to choose:

Option A	\$1,000,000 lump sum paid today
Option B	\$1,000 every week, guaranteed for life

- **Object:** two streams with different amounts and dates.
- **Method:** move every cash flow to one declared comparison date.
- **Decision:** compare present values, then state the assumptions.

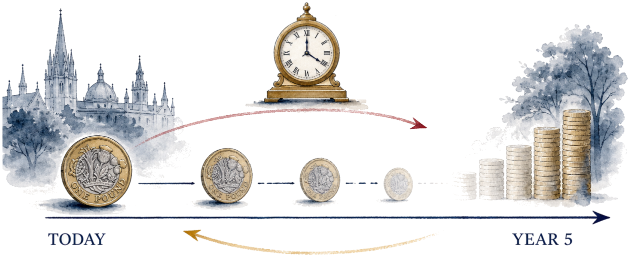
Adding future payments without discounting mixes values from different dates.

Day 1 roadmap: three questions about dated cash flows

1. **Dates:** how can values at different dates be compared?
2. **Value:** why do present value and NPV provide a defensible comparison?
3. **Patterns:** when can a cash-flow list be compressed into an annuity, perpetuity, or loan formula?

NPV = net present value: value today of benefits minus value today of costs. Day 2 will examine which interest-rate quotation supplies the discount factors.

£100 today and £100 later are different economic objects



- **Opportunity:** money held today can earn a return before year 2.
- **Purchasing power:** future prices may differ from today's prices.
- **Uncertainty:** a promised future payment may not be certain.

Day 1 builds dated cash flows; Day 2 adds rate conventions, inflation, maturity, and market prices.

By the end of Day 1, you should be able to value the cash-flow pattern

- **Represent** a decision as dated, signed cash flows.
- **Calculate** future value, present value, and NPV at a declared flat rate.
- **Explain** the Law of One Price and construct a simple arbitrage table.
- **Recognise** perpetuities, annuities, growing streams, and equal-payment loans.
- **Audit** timing, signs, units, and formula assumptions before interpreting a result.

PV = present value; FV = future value; NPV = net present value. Rates are effective per timeline period unless stated otherwise.

Every valuation begins with five declarations

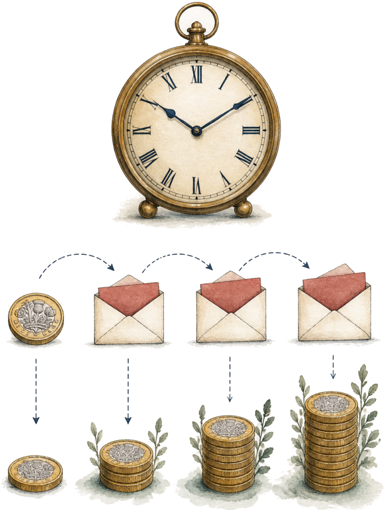
Declaration	Question to answer before calculation
Perspective	Who receives positive cash flows and pays negative cash flows?
Dates	When exactly does each cash flow occur?
Period	Does one period mean a year, month, quarter, or day?
Units	Which currency, and are amounts nominal or real?
States	Is payment certain, risky, indexed, optional, or contingent?

The main Day 1 calculations use certain sterling cash flows and a declared effective rate per period.

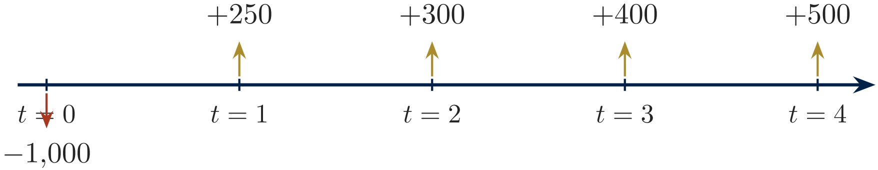
Move value through time

How can values at different dates be compared?

Build a timeline, compound forward, and discount backward to one common date.

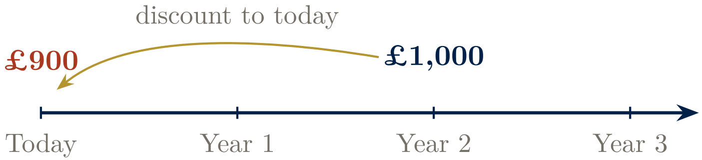


A timeline is the grammar of valuation



- The sign records the decision maker's perspective, not whether a cash flow is "good."
- The spacing defines the period used by the interest rate.
- The timeline reveals whether a formula describes the actual contract.

Rule 1: compare or add values only at a common date



- "£900 + £1,000" is not a valuation when the amounts occur at different dates.
- Move both to today, both to year 2, or any other declared common date.
- Consistent market rates preserve the ranking whichever common date is chosen.

Interest is an exchange rate between dates

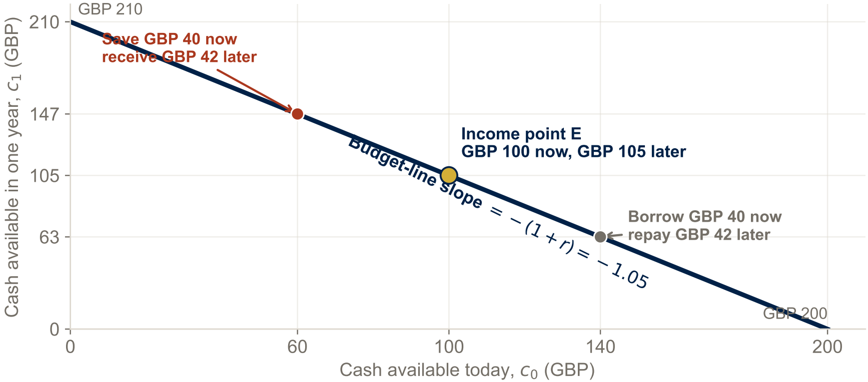
If the one-year annual effective rate is r, the market exchange is

$$\text{£}1 \text{ today} \quad \longleftrightarrow \quad \text{£}(1+r) \text{ in one year.}$$

r: one-year annual effective rate · $1+r$: gross accumulation factor

- Lending moves purchasing power forward: give up one pound now and receive $1+r$ pounds later.
- Borrowing reverses the exchange: receive now and promise $1+r$ pounds later.
- A rate is incomplete without its horizon, risk, currency, and quotation convention.

The same rate prices saving and borrowing across two dates



GBP = pound sterling · c_0: cash available today · c_1: cash available in one year · E: initial income point

At 5%, moving £40 across the two dates changes one-year resources by £42. The slope is $-(1+r)$: the interest rate is the relative price of dated cash.

Rule 2: compounding moves value forward

$$\mathrm{FV}_1 = C_0(1+r), \qquad \mathrm{FV}_T = C_0(1+r)^T.$$

FV_T: future value at date T · C_0: cash flow today · T: number of compounding periods

- Each period earns interest on the balance carried into that period.
- The exponent counts how many market exchanges separate today from the target date.
- The rate period must match the timeline period.

Appendix A2 · Derivation Python P1 · Time value

One-period future value: the interest earned is explicit

Deposit £8,000 for one year at a 4.5% annual effective rate.

$$\mathrm{FV}_1 = 8{,}000(1.045) = \pounds 8{,}360.$$

Opening balance	£8,000
Interest $= 0.045 \times 8{,}000$	£360
Closing balance	£8,360

For one period, simple and compound interest coincide; their difference appears only after interest itself begins to earn interest.

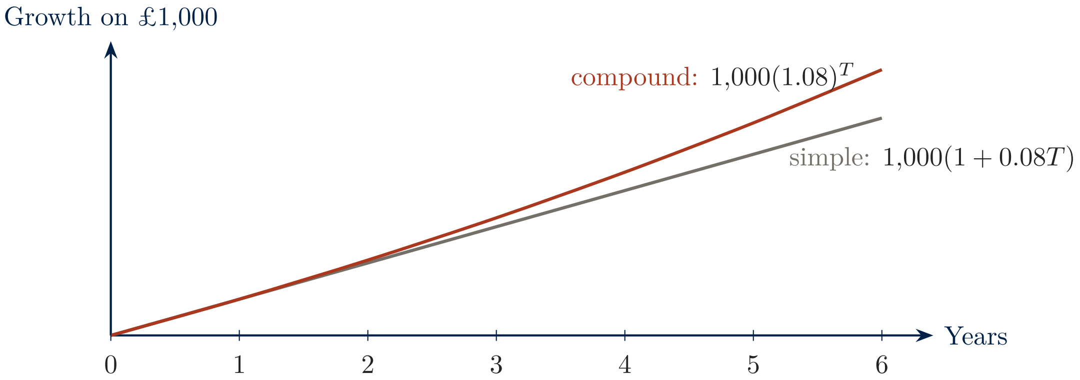
Multi-period growth is multiplicative

The same £8,000 remains invested for three years at 4.5%:

$$\text{FV}_3 = 8{,}000(1.045)^3 = \pounds 9{,}129.33.$$

Year	Opening balance	Interest	Closing balance
1	8,000.00	360.00	8,360.00
2	8,360.00	376.20	8,736.20
3	8,736.20	393.13	9,129.33

Simple interest omits interest on interest



At 8% for six years, compounding adds £586.87; simple interest adds only £480.

The same equation can reveal an unknown rate or horizon

From $\mathrm{FV}_T = C_0(1+r)^T$,

$$r = \left(\frac{\mathrm{FV}_T}{C_0}\right)^{1/T} - 1, \qquad T = \frac{\ln(\mathrm{FV}_T/C_0)}{\ln(1+r)}.$$

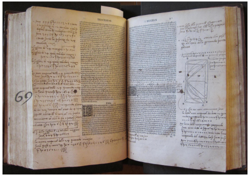
- Doubling £1,000 to £2,000 in nine years requires $r = 2^{1/9} - 1 = 8.01\%$.
- At 6%, doubling takes $T = \ln 2 / \ln 1.06 = 11.90$ years.
- Logs convert repeated multiplication into a solvable linear relation in T.

How many years do I need?

Tell me: how much time would my investment take to double, given the current return?

Who is this guy?

Father of Accounting: Luca Pacioli



Rule of 72 and Rule of 69.3

The rule of 72 is a simple method to determine how long an investment takes to double, given a fixed annual interest rate. Divide 72 by the annual rate of return:

$$t = \frac{\ln 2}{\ln(1 + r/100)} \approx \frac{72}{r} \tag{1}$$

Rule of 70

The rule of 70 is used to determine the number of years it takes for a variable to double or halve, by dividing 70 by the growth or decay rate. It is generally used to ask how long an investment takes to double or halve, given the annual rate of return.

$$t = \frac{70}{r} \tag{2}$$

Rule 3: discounting moves value backward

Compounding and discounting are inverse operations:

$$\mathrm{FV}_T = C_0(1+r)^T \qquad \Longleftrightarrow \qquad \mathrm{PV}_0 = \frac{C_T}{(1+r)^T}.$$

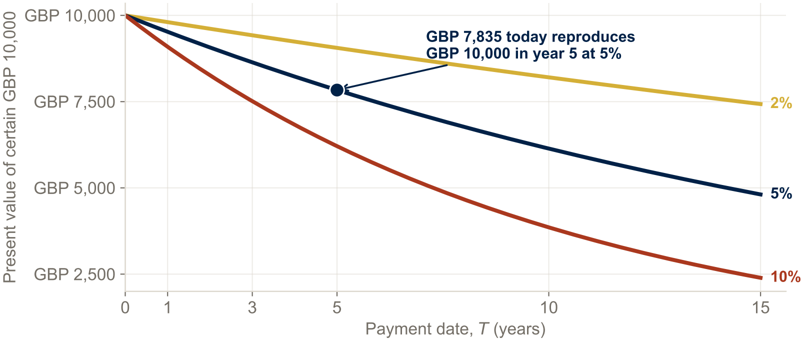
C_T: payment at date T · PV_0: equivalent value at date 0

- Discounting asks how much invested today would reproduce the future payment.
- A higher opportunity return means less money is needed today.
- A later payment is discounted through more periods.

Appendix A4 · Inverse operation and date invariance

One future payment: ask for the replicating amount today

$$\mathrm{PV}_0 = \frac{10{,}000}{(1.05)^5} = £7{,}835.26.$$



A later payment or a higher comparable return lowers present value because less money is needed today to reproduce the same dated payoff.

An uneven stream is valued one date at a time

For dated cash flows $C_0, \ldots, C_3$ under a flat effective rate $r = 6\%$:

$$\mathrm{PV}_0 = \sum_{t=0}^{3} \frac{C_t}{(1+r)^t} = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3}.$$

C_t: signed cash flow at date t · $t = 0$ initial outlay is not discounted

Date t	**Cash flow** C_t	**Discount factor** $d_t = 1.06^{-t}$	**Present value** (£)
0	−£1,000.00	1.0000	−1,000.00
1	+£300.00	0.9434	+283.02
2	+£450.00	0.8900	+400.50
3	+£500.00	0.8396	+419.81
Total	**+£250.00**	*undiscounted sum*	**+£103.33**

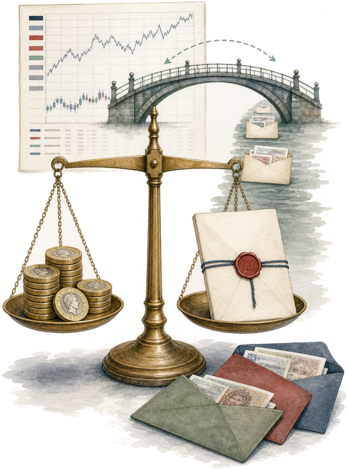
Value additivity requires discounting each dated claim to $t = 0$ before summing. The simple undiscounted sum (+£250) ignores the opportunity cost of time.

Appendix A8 · Vector form and checks

Present value and NPV

Why is present value a defensible comparison?

Use replication, opportunity cost, and the NPV rule to connect a calculation to an available market alternative.



A discount factor is the price of one future pound

$$d_t = \mathrm{PV}_0(£1 \text{ at } t) = \frac{1}{(1+r)^t}, \qquad \mathrm{PV}_0(C_t) = d_t C_t.$$

d_t: date-0 discount factor for maturity t · C_t: cash flow at date t

Building block	Formula	Economic meaning
Discount factor d_t	$d_t = (1+r)^{-t}$	Price today of £1 delivered at date t
Single cash flow	$\mathrm{PV}_0(C_t) = d_t C_t$	Scale unit price by cash flow size
Cash-flow stream	$\mathrm{PV}_0 = \sum_{t=0}^{T} d_t C_t$	Value additivity across dated claims

- **Price vs rate:** The discount factor d_t is the market price; r is a quoting convention.
- **Vector form:** $\mathrm{PV}_0 = \boldsymbol{d}^\top \boldsymbol{C}$; expand the product as $d_0 C_0 + \cdots + d_T C_T$.

Discount factors convert dated cash flows into date-0 pounds before combining.

The Law of One Price disciplines valuation

If two traded portfolios deliver identical cash flows in every relevant date and state,

$$\boldsymbol{C}^A = \boldsymbol{C}^B \qquad \Longrightarrow \qquad P(A) = P(B)$$

$\boldsymbol{C}^A, \boldsymbol{C}^B$: complete dated, state-contingent cash-flow vectors · $P(A), P(B)$: current market prices

in an arbitrage-free market.

- Otherwise, buy the cheaper payoff and sell the more expensive copy.
- "Same expected payoff" is not enough; the dated state-by-state payoff must match.
- Frictions limit real trades but do not remove the benchmark logic.

Replication derives the price of a certain claim

A one-year deposit paying 4% can reproduce a certain £1,000 invoice:

$$P^* = \frac{1{,}000}{1.04} = £961.54.$$

P^*: no-arbitrage replicating price under the stated borrowing and lending opportunity

Discounting is the numerical expression of a replicating trade when borrowing and lending span the future payment.

If the claim is underpriced, the future cash flows finance the trade

Suppose the certain £1,000 claim trades at £950 while borrowing for one year costs 4%.

$+950 - 950 = 0$ $\qquad$ $+1{,}000 - 988 = +12$

today → year 1

- Borrow £950 today and use it to buy the claim.
- In one year the claim pays £1,000; the loan requires £988.
- Net cash flow: zero today and £12 in one year.

The detailed trade table checks every date rather than reporting only the final profit.

Appendix A5 · Full arbitrage table

If the claim is overpriced, reverse the positions

Suppose the same claim trades at £975.

$+975 - 975 = 0$ $\qquad\qquad$ $+1{,}014 - 1{,}000 = +14$

today $\longrightarrow$ year 1

- Short or sell the claim for £975 today.
- Lend the proceeds at 4%; the deposit becomes £1,014.
- Pay the promised £1,000 and retain £14 in one year.

Short-sale constraints, bid-ask spreads, taxes, and funding limits can prevent the literal trade; the pricing comparison remains informative.

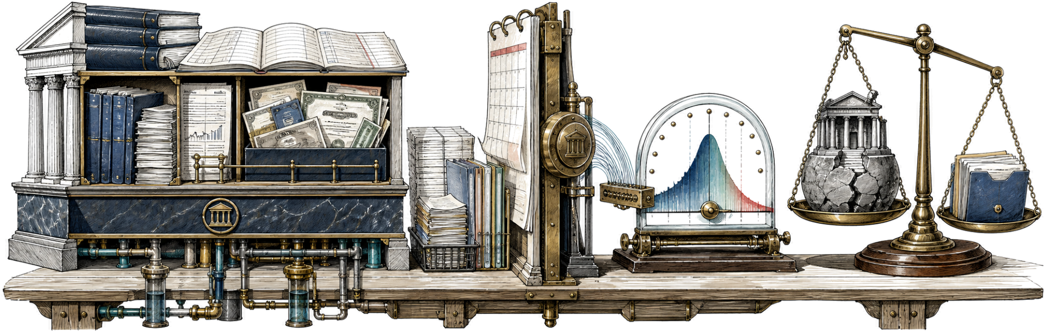
The opportunity cost of capital is a market alternative

- It is the return available on an alternative with comparable timing and risk.
- It is not chosen because it is the firm's customary hurdle rate or the manager's target.
- Certain contractual payments can begin with near-risk-free market prices.
- Risky operating cash flows require a risk-adjusted comparison; equal expected cash flows need not have equal values.

> Day 1 uses certainty to establish the valuation engine. Day 4's CAPM material will address the required return on systematic risk.

CAPM = capital asset pricing model; it links required expected return to systematic market risk under its assumptions.

Project Valuation



Project valuation follows four explicit steps

1. **Forecast** incremental cash flows, including timing and sign.
2. **Identify** a market opportunity cost matching maturity and risk.
3. **Discount** every cash flow to one comparison date.
4. **Decide** by comparing present value of benefits with present value of costs.

This sequence follows the corporate-finance textbooks but makes the market comparison and computational representation explicit.

Worked project: an energy retrofit changes operating cash flows

A warehouse retrofit costs £420,000 at $t = 0$; annual energy savings are discounted at opportunity cost $r = 7\%$:

$$\mathrm{PV}_0(\text{Savings}) = \sum_{t=1}^{4} \frac{C_t}{(1+r)^t} = \frac{£120\text{k}}{1.07} + \frac{£140\text{k}}{1.07^2} + \frac{£160\text{k}}{1.07^3} + \frac{£100\text{k}}{1.07^4} = £441{,}328.$$

C_t: nominal year-end cash savings · $d_t = 1.07^{-t}$: discount factor at opportunity cost $r = 7\%$

Year t	**Cash saving** C_t	**Discount factor** $d_t = 1.07^{-t}$	**Present value** (£)
1	£120,000	0.9346	£112,150
2	£140,000	0.8734	£122,281
3	£160,000	0.8163	£130,608
4	£100,000	0.7629	£76,290
Total	**£520,000**	*undiscounted sum*	**£441,328**

Savings PV of £441,328 exceeds the £420,000 outlay, so NPV = +£21,328.

NPV measures value created relative to the alternative

$$\text{NPV}_0 = \text{PV}_0(\text{benefits}) - \text{PV}_0(\text{costs}).$$

For the retrofit,

$$\text{NPV}_0 = 441{,}328 - 420{,}000 = £21{,}328.$$

- Accept an independent project when NPV is positive.
- Positive NPV means the project buys its dated payoff more cheaply than the comparison asset.
- The numerical result remains conditional on forecast cash flows and the 7% opportunity cost.

Appendix A4 · Comparison-date invariance

Mutually exclusive alternatives require incremental comparison

Suppose two retrofit designs cannot both be installed:

Design	PV benefits	Cost today	NPV
Standard	441,328	420,000	21,328
Advanced	520,000	485,000	35,000

- Both pass the independent accept/reject test.
- If capacity permits only one, choose the advanced design because its NPV is larger.
- Comparing IRRs or payback periods can rank mutually exclusive scale and timing patterns incorrectly.

IRR = internal rate of return, a discount rate that sets NPV to zero; payback period = time until cumulative undiscounted inflows recover the initial outlay.

Value additivity turns a contract into computable pieces

$$\mathrm{PV}_0(A+B)=\mathrm{PV}_0(A)+\mathrm{PV}_0(B), \qquad \mathrm{PV}_0=\boldsymbol{d}^\top\boldsymbol{C}.$$

$\boldsymbol{C}$: aligned vector of dated cash flows · $\boldsymbol{d}$: matching discount-factor vector

- Additivity explains why each payment can be priced separately and summed.
- The dot product is the computational form of the same economic principle.
- A dimension mismatch or date misalignment is a contract error, even if software returns a number.

Appendix A8 · Vector checks Python P2 · NPV arrays

IRR is a break-even rate; NPV remains the decision rule

For a cash-flow stream, an internal rate of return r^* solves

$$0 = \sum_{t=0}^{T} \frac{C_t}{(1 + r^*)^t}.$$

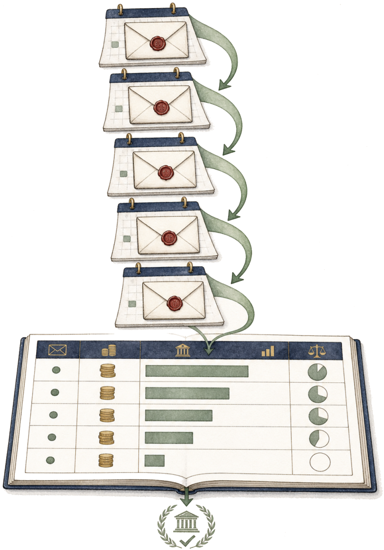
- For one initial outflow followed by positive inflows, IRR often gives a useful return summary.
- Accepting when r^* exceeds the opportunity cost is equivalent to positive NPV only under standard cash-flow patterns.
- Multiple sign changes can produce multiple IRRs; mutually exclusive projects can be ranked incorrectly.

IRR = internal rate of return; r^* is a root that makes NPV zero. Later we use the same root-finding idea for a financing threshold without treating IRR as a universal decision rule.

Repeated cash flows and loans

When can a long cash-flow list be compressed safely?

Recognise the timeline first; only then use perpetuities, annuities, growth formulas, and loan identities.

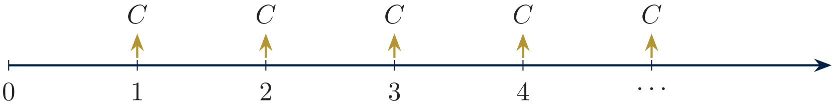


Repeated cash flows have recognisable shapes

Pattern	Timeline	Typical application
Perpetuity	level payments forever	endowment or preferred share
Ordinary annuity	level payments at $1, \ldots, T$	loan or lease payments
Annuity due	level payments at $0, \ldots, T-1$	rent paid in advance
Growing stream	payments grow at constant g	indexed cost or contribution plan

A shortcut is valid only when payment amount, spacing, first date, horizon, and rate period match its cash-flow pattern.

A perpetuity pays a level amount forever



$$\mathrm{PV}_0 = \frac{C}{r}.$$

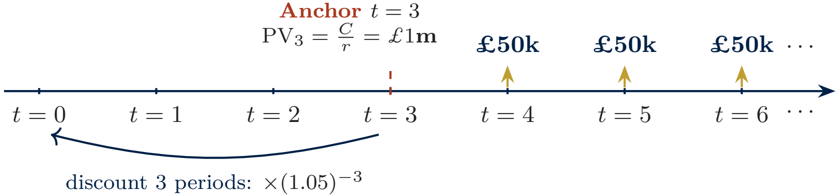
First payment at $t = 1$; level payment C; constant effective rate $r > 0$

- A £80,000 annual endowment distribution requires £2.0m at 4%.
- The fund earns £80,000 each year and distributes the interest while preserving principal.

Appendix A9 · Geometric-series derivation

A delayed perpetuity is valued one date before it begins

A stream of £50,000 annual payments begins at the end of year 4; $r = 5\%$:



Step 1: Value at anchor date $t = 3$	Step 2: Discount back to date 0
$\mathrm{PV}_3 = \dfrac{C}{r} = \dfrac{£50{,}000}{0.05} = £1{,}000{,}000$	$\mathrm{PV}_0 = \dfrac{\mathrm{PV}_3}{(1+r)^3} = \dfrac{£1{,}000{,}000}{(1.05)^3} = £863{,}838$
PV_3: date-3 lump sum of cash flows $t \geq 4$	PV_0: date-0 present value discounted 3 periods

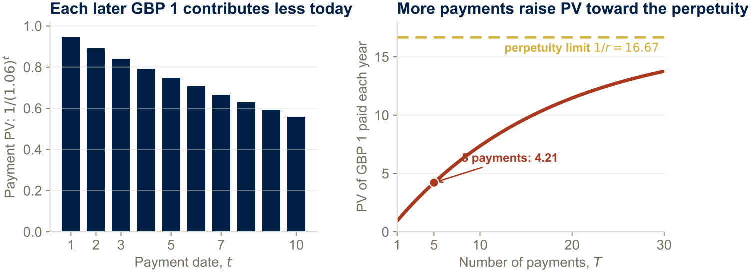
Perpetuity formula values cash flows at date $k - 1$; discounting 4 periods double-counts the delay.

An ordinary annuity is a finite portfolio of dated claims

A level payment C arrives at dates $1, \ldots, T$:

$$\mathrm{PV}_0 = C \sum_{t=1}^{T} (1+r)^{-t}.$$

PV_0: date-0 present value · C: level payment at each date $t = 1, \ldots, T$ · r: discount rate



At 6%, later payments carry smaller present-value weights. Extending the horizon raises annuity value toward, but never above, the perpetuity limit C/r.

The annuity factor compresses the finite sum

$$\mathrm{PV}_0 = C\frac{1-(1+r)^{-T}}{r} = C \times \mathrm{PVAF}(r, T).$$

PVAF = present-value annuity factor · C: level end-of-period payment · T: number of payments · r: effective rate per payment period

- The first payment is one period from the valuation date.
- Payment frequency and rate frequency must agree.
- At $r = 6\%$, five annual payments of £2.15m have PV £9.0566m.

An annuity equals one perpetuity minus its unwanted tail

$$\underbrace{\frac{C}{r}}_{\text{payments } 1,2,\ldots} \quad - \quad \underbrace{\frac{C}{r(1+r)^T}}_{\text{payments } T+1, T+2,\ldots} \quad = C\frac{1-(1+r)^{-T}}{r}.$$

- The first perpetuity supplies all desired annuity payments and an unwanted tail.
- The delayed perpetuity removes precisely the payments after date T.
- As $T \to \infty$, the tail's present value vanishes and the annuity approaches C/r.

Moving every payment one period earlier creates an annuity due

$$\mathrm{PV}_0^{\text{due}} = (1+r)\,\mathrm{PV}_0^{\text{ordinary}} = (1+r)C\,\frac{1-(1+r)^{-T}}{r}.$$

Pattern	Payment dates	Value relation
Ordinary annuity	$1,\ldots,T$	base formula
Annuity due	$0,\ldots,T-1$	ordinary value $\times(1+r)$

Future value accumulates each contribution to the target date

For T end-of-period contributions,

$$\mathrm{FV}_T = C\frac{(1+r)^T - 1}{r}.$$

- The payment at $t = 1$ compounds for $T - 1$ periods.
- The final payment at $t = T$ earns no interest before the target date.
- Investing £5,000 annually for ten years at 5% produces £62,889.46.

The future-value factor is the present-value annuity factor carried forward by $(1+r)^T$.

A growing perpetuity separates discounting from cash-flow growth

If the first payment at $t = 1$ is C_1 and payments grow at rate g,

$$\mathrm{PV}_0 = \frac{C_1}{r - g}, \qquad r > g.$$

$C_t = C_1(1+g)^{t-1}$ · r and g use the same period and basis

- A 2% indexed £80,000 first distribution requires £4.0m when $r = 4\%$.
- Growth reduces the net rate at which distant payments lose present value.
- When $g \geq r$, the infinite discounted sum does not converge under these assumptions.

A growing annuity has a finite horizon

For payments $C_1, C_1(1+g), \ldots, C_1(1+g)^{T-1}$,

$$\mathrm{PV}_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right], \qquad r \neq g.$$

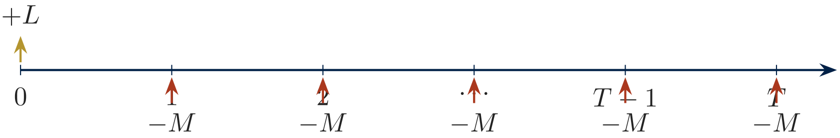
- Use this for a finite indexed lease, contribution plan, or cost stream.
- With $C_1 = £10,000$, $g = 3\%$, $r = 7\%$, $T = 8$, PV is £65,682.
- At $r = g$, each discounted payment equals $C_1/(1+r)$; the limiting value is $TC_1/(1+r)$.

A loan exchanges principal today for payments later

Borrow L today and repay M at the end of each of T periods:

$$L = M\,\frac{1-(1+r)^{-T}}{r}.$$

L: loan principal received at date 0 · M: equal end-of-period payment · T: number of payments



Solve for the equal payment, not by trial and error

$$M = L\frac{r}{1-(1+r)^{-T}}.$$

Borrow £20,000 for four years at 6% with annual payments:

$$M = 20{,}000\frac{0.06}{1-(1.06)^{-4}} = £5{,}771.83.$$

- Total paid is £23,087.32, but payments occur at different dates.
- Their present value at the contractual rate is exactly the £20,000 principal.

Appendix A15 · Payment algebra | Python P3 · Loan schedule

Each payment contains interest and principal repayment

Let B_{t-1} be the opening balance before payment t:

$$\text{Interest}_t = rB_{t-1}, \qquad \text{Principal}_t = M - rB_{t-1}, \qquad B_t = B_{t-1}(1+r) - M.$$

Year	Opening	Interest	Principal	Closing
1	20,000.00	1,200.00	4,571.83	15,428.17
2	15,428.17	925.69	4,846.14	10,582.03

The outstanding balance is the PV of remaining payments

After payment k, with $T - k$ payments left,

$$B_k = M \frac{1 - (1 + r)^{-(T-k)}}{r}.$$

- The backward-looking recursion and forward-looking PV must produce the same balance.
- This identity is a powerful check on an amortisation schedule.
- Settlement fees or prepayment penalties are additional dated cash flows, not part of the annuity identity.

Non-annual cash flows require period alignment

For a 6% APR compounded monthly, the monthly effective rate is

$$r_m = \frac{0.06}{12} = 0.5\%.$$

A five-year monthly loan therefore uses $T = 60$, not $T = 5$:

$$M = L\frac{0.005}{1 - (1.005)^{-60}}.$$

- Cash-flow period: one month.
- Rate period: one month.
- Number of periods: 60 months.

Day 1 leaves a reusable cash-flow valuation engine

1. Draw dated, signed cash flows from one perspective.
2. Choose one comparison date and one rate period.
3. Compound or discount each cash flow to that date.
4. Use a shortcut only when the payment pattern matches its assumptions.
5. Interpret PV or NPV together with units, timing, and omitted risks.

> Day 2 asks which rate quotation and maturity-specific market price should be used in that engine.

References and data sources

- Berk, J. and DeMarzo, P., *Corporate Finance*, Ch. 4, “The Time Value of Money.”
- Brealey, R., Myers, S., Allen, F. and Edmans, A., *Principles of Corporate Finance*, Ch. 2, “How to Calculate Present Values.”
- Loto-Québec, *Gagnant à vie*, 2025 winner announcement; used only to frame a dated-cash-flow decision.

- All numerical examples were independently checked; constructed cases are identified on the relevant slide.

Course materials and permitted use

- Materials are provided for students enrolled in Computational Finance & FinTech.
- Do not reproduce, redistribute, or adapt the slides without asking the author first.
- Cite the named books, institutions, datasets, and papers when using their material.
- Course page: fatih.ai/computationalfinance

Technical appendix map

← **Return to the Day 1 synthesis**

Practical map

Appendix A1: Timeline and sign audit

1. Declare the decision maker's perspective.
2. List exact payment dates before replacing them with period numbers.
3. Record receipts as positive and payments as negative.
4. Confirm that one rate period equals one timeline interval.
5. Mark uncertain, indexed, or optional cash flows separately.

← **Return to the timeline**

Appendix map

Appendix A2: Deriving compound growth

$$C_1 = C_0(1+r),$$

$$C_2 = C_1(1+r) = C_0(1+r)^2,$$

$$C_3 = C_2(1+r) = C_0(1+r)^3.$$

Repeating the same substitution for T periods gives

$$C_T = C_0(1+r)^T.$$

The exponent counts repeated multiplication by the gross return, not calendar years unless the rate period is annual.

← **Return to compounding**

Appendix map

Appendix A3: Solving for an unknown rate or horizon

Starting from $\mathrm{FV}_T = C_0(1+r)^T$:

$$\frac{\mathrm{FV}_T}{C_0} = (1+r)^T,$$

$$\left(\frac{\mathrm{FV}_T}{C_0}\right)^{1/T} = 1+r,$$

$$r = \left(\frac{\mathrm{FV}_T}{C_0}\right)^{1/T} - 1.$$

Taking logs instead gives

$$T = \frac{\ln(\mathrm{FV}_T / C_0)}{\ln(1+r)}.$$

← **Return to unknown rate or horizon**

Appendix map

Appendix A4: Discounting and comparison-date invariance

From $C_T = C_0(1+r)^T$, divide by $(1+r)^T$:

$$C_0 = \frac{C_T}{(1+r)^T}.$$

If every dated cash flow is moved to date k at the same flat rate r, then for alternative j

$$V_k^j = \sum_t C_t^j (1+r)^{k-t} = (1+r)^k V_0^j.$$

Thus $V_0^A > V_0^B$ if and only if $V_k^A > V_k^B$. For $t < k$, this comparison assumes interim cash flows can be reinvested or financed at the same rate.

Appendix A5: Underpriced claim — full arbitrage table

Trade	$t = 0$	$t = 1$
Borrow £950 at 4%	+950	−988
Buy certain £1,000 claim	−950	+1,000
Net cash flow	**0**	**+12**

The trade has no initial cost and a positive certain future payoff under the stated frictionless assumptions.

← **Return to underpricing**

Appendix map

Appendix A6: Overpriced claim — reverse-trade audit

Trade	$t = 0$	$t = 1$
Short/sell claim	+975	−1,000
Lend £975 at 4%	−975	+1,014
Net cash flow	**0**	**+14**

In a market with bid-ask spreads, use executable borrowing, lending, buying, and selling prices in the audit.

← **Return to overpricing**

Appendix map

Appendix A7: Equal expected payoff does not imply equal price

Claim	Good state ($p = 0.5$)	Bad state ($p = 0.5$)	Expected payoff
A: certain	100	100	100
B: state-dependent	200	0	100

Claim A pays in both states; Claim B pays only in the good state. Their prices depend on state prices and risk exposure, not only the arithmetic mean payoff.

← **Return to opportunity cost**

Appendix map

Appendix A8: Vector implementation and validation identities

$$\mathrm{PV} = \boldsymbol{d}^\top \boldsymbol{C} = \sum_{t=0}^{T} d_t C_t.$$

- Shape check: $\mathrm{len}(\boldsymbol{d}) = \mathrm{len}(\boldsymbol{C})$.
- Date check: discount factors correspond to the same ordered dates as cash flows.
- Unit check: rates are decimals; cash-flow units are consistent.
- Identity check: $d_0 = 1$; under positive certain rates, $0 < d_t \leq 1$.
- Formula check: explicit cash-flow sum equals any valid annuity shortcut.

$\mathrm{len}(\cdot)$: number of entries in a vector; $d_0 = 1$ because a date-0 pound is already at the valuation date.

← **Cash-flow sum** ← **Additivity**

Appendix map

Appendix A9: Perpetuity geometric-series derivation

Let $x = (1+r)^{-1}$, so $0 < x < 1$ when $r > 0$:

$$\begin{aligned} \mathrm{PV}_0 &= Cx + Cx^2 + Cx^3 + \cdots, \\ x\,\mathrm{PV}_0 &= Cx^2 + Cx^3 + Cx^4 + \cdots, \\ (1-x)\,\mathrm{PV}_0 &= Cx, \\ \mathrm{PV}_0 &= \frac{Cx}{1-x} = \frac{C}{r}. \end{aligned}$$

← **Return to perpetuities**

Appendix map

Appendix A10: Delayed perpetuity timing

If the first payment C occurs at date $k+1$, the perpetuity is worth C/r at date k:

$$\mathrm{PV}_k = \frac{C}{r}.$$

Move that value from date k to date 0:

$$\mathrm{PV}_0 = \frac{C/r}{(1+r)^k}.$$

The exponent is k, not $k+1$, because C/r is already the value one period before the first payment.

← **Return to delayed perpetuities**

Appendix map

Appendix A11: Ordinary-annuity derivation

Let $x = (1 + r)^{-1}$:

$$\begin{aligned}
\mathrm{PV}_0 &= C(x + x^2 + \cdots + x^T), \\
x\,\mathrm{PV}_0 &= C(x^2 + x^3 + \cdots + x^{T+1}), \\
(1 - x)\,\mathrm{PV}_0 &= C(x - x^{T+1}), \\
\mathrm{PV}_0 &= C\,\frac{x(1 - x^T)}{1 - x} = C\,\frac{1 - (1 + r)^{-T}}{r}.
\end{aligned}$$

← **Return to the annuity factor**

Appendix map

Appendix A12: Annuity-due and future-value identities

Shifting each ordinary-annuity payment one period earlier gives

$$\mathrm{PV}_0^{\mathrm{due}} = (1+r)\,\mathrm{PV}_0^{\mathrm{ordinary}}\,.$$

Carrying the ordinary annuity's PV to date T gives

$$\begin{aligned}\mathrm{FV}_T &= (1+r)^T C\,\frac{1-(1+r)^{-T}}{r}\\ &= C\,\frac{(1+r)^T-1}{r}.\end{aligned}$$

← **Return to payment timing**

Appendix map

Appendix A13: Growing-perpetuity derivation and convergence

$$\begin{aligned}
\mathrm{PV}_0 &= \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \cdots \\
&= \frac{C_1}{1+r}\left[1 + q + q^2 + \cdots\right], \qquad q = \frac{1+g}{1+r}.
\end{aligned}$$

The series converges when $|q| < 1$; for ordinary positive rates this requires $r > g$. Then

$$\mathrm{PV}_0 = \frac{C_1}{1+r}\,\frac{1}{1-q} = \frac{C_1}{r-g}.$$

← **Return to growing perpetuities**

Appendix map

Appendix A14: Growing-annuity derivation and the $r = g$ case

With $q = (1+g)/(1+r)$,

$$\mathrm{PV}_0 = \frac{C_1}{1+r}\sum_{t=0}^{T-1} q^t = \frac{C_1}{1+r}\,\frac{1-q^T}{1-q}.$$

Simplifying for $r \neq g$ gives

$$\mathrm{PV}_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right].$$

If $r = g$, then $q = 1$ and every discounted payment equals $C_1/(1+r)$, hence $\mathrm{PV}_0 = TC_1/(1+r)$.

← **Return to growing annuities**

Appendix map

Appendix A15: Solving the equal-payment loan

The principal equals the PV of payments:

$$L = M\frac{1-(1+r)^{-T}}{r}.$$

Divide both sides by the annuity factor:

$$M = \frac{L}{[1-(1+r)^{-T}]/r} = L\frac{r}{1-(1+r)^{-T}}.$$

Verification: substitute the calculated payment back into the PV equation; the result should recover L up to rounding.

← **Return to loan payments**

Appendix map

Appendix A16: Complete amortisation audit

Year	Opening	Payment	Interest	Principal	Closing
1	20,000.00	5,771.83	1,200.00	4,571.83	15,428.17
2	15,428.17	5,771.83	925.69	4,846.14	10,582.03
3	10,582.03	5,771.83	634.92	5,136.91	5,445.12
4	5,445.12	5,771.83	326.71	5,445.12	0.00

Rounding the displayed payment can leave a few pence in the final balance; production schedules retain full precision internally.

← **Return to amortisation**

Appendix map

Appendix A17: General non-annual period conversion

If an EAR is known and there are m equal periods per year,

$$r_p = (1 + r_{\text{EAR}})^{1/m} - 1.$$

If a nominal APR q_m compounded m times is known,

$$r_p = \frac{q_m}{m}.$$

Over Y years, use $T = mY$ payment periods. For irregular dates, use exact year fractions and discount factors rather than forcing an annuity formula.

← **Return to period alignment**

Appendix map

Python practical map

1. P1 · Compound and discount one cash flow →
2. P2 · Calculate NPV with aligned arrays →
3. P3 · Build a loan amortisation schedule →

Use the supplied Day 1 notebook and frozen CSV data; rates are decimals in code and monetary units are declared in each task.

Technical appendix Lecture synthesis

Practical P1: Compound forward, then discount back

- Store the declared cash flow, rate, and horizon separately.
- Compute the year-3 value and recover the original date-0 value.

```
# Inputs use decimals: 4.5% becomes 0.045.
cash_today = 8000.0
annual_rate = 0.045
years = 3

future_value = cash_today * (1 + annual_rate) ** years
present_value = future_value / (1 + annual_rate) ** years
print(round(future_value, 2), round(present_value, 2))
```

`=` assigns a value; `**` means exponentiation; `round(x, 2)` displays two decimal places. Expected output: `9129.33 8000.0`.

← **Return to compounding**

Practical map

Practical P2: NPV is a dot product of aligned arrays

- Create one array for dates and one for signed cash flows.
- Build discount factors in the same order, then multiply and add.

```
import numpy as np

times = np.array([0, 1, 2, 3])
cash_flows = np.array([-1000.0, 300.0, 450.0, 500.0])
discount_factors = 1 / (1 + 0.06) ** times
npv = np.dot(discount_factors, cash_flows)
print(round(npv, 2))
```

NumPy stores numerical arrays; `np.dot(a, b)` multiplies matching entries and sums them. Expected NPV: `103.33`.

← **Return to value additivity**

Practical map

Practical P3: Build and audit a loan schedule

- Calculate the contractual payment once at full precision.
- Update interest, principal repayment, and balance period by period.

```
import pandas as pd

loan, rate, years = 20000.0, 0.06, 4
payment = loan * rate / (1 - (1 + rate) ** (-years))
balance, rows = loan, []

for year in range(1, years + 1):
    interest = rate * balance
    principal = payment - interest
    balance = balance - principal
    rows.append([year, interest, principal, balance])

schedule = pd.DataFrame(rows, columns=["year", "interest", "principal", "balance"])
```

pandas `DataFrame` makes a labelled table; a `for` loop repeats the indented update; `append` stores each row. The final balance should be numerically close to zero.

← **Return to loan payments**

Practical map