The Price of Time

Present value, interest rates, and computational decisions

Summer 2026

Computational Finance / FinTech Summer School

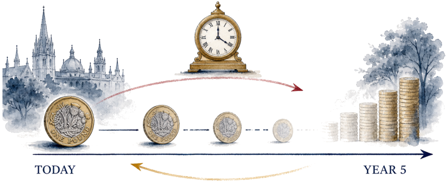
Day 1 | final-year undergraduate / introductory master's level

Day 1 roadmap: six questions, one valuation engine

1. **Dates:** how can values at different dates be compared?
2. **Price:** why is present value a defensible market comparison?
3. **Patterns:** when can a cash-flow list be compressed safely?
4. **Rates:** when do two percentages represent the same price?
5. **Curve:** which market rate prices each payment date?
6. **Decision:** can the complete engine support a recommendation?

NPV = net present value: value today of benefits minus value today of costs. The lecture builds one valuation engine; each section removes one reason that two cash-flow plans might be compared incorrectly.

£100 today and £100 later are different economic objects



- **Opportunity:** money held today can earn a return before year 2.
- **Purchasing power:** future prices may differ from today's prices.
- **Uncertainty:** a promised future payment may not be certain.

Time value solves the first problem. Inflation and risk determine which cash flows and market prices belong in that solution.

By the end, you should be able to defend the number

- **Represent** a financial decision as dated, signed cash flows.
- **Calculate** future value, present value, NPV, annuities, and loan payments.
- **Translate** APR, EAR, continuous, nominal, and real rates consistently.
- **Select** discount factors whose maturity and risk match each cash flow.
- **Diagnose** common modelling errors before they become numerical errors.
- **Communicate** a recommendation together with assumptions and thresholds.

PV = present value; FV = future value; APR = annual percentage rate; EAR = effective annual rate.

Every valuation begins with five declarations

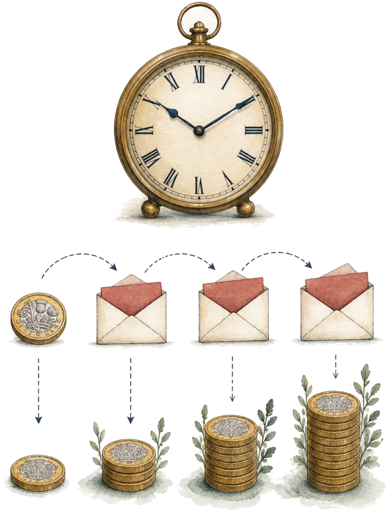
Declaration	Question to answer before calculation
Perspective	Who receives positive cash flows and pays negative cash flows?
Dates	When exactly does each cash flow occur?
Period	Does one period mean a year, month, quarter, or day?
Units	Which currency, and are amounts nominal or real?
States	Is payment certain, risky, indexed, optional, or contingent?

The calculations in the first half assume certain sterling cash flows unless a slide states otherwise.

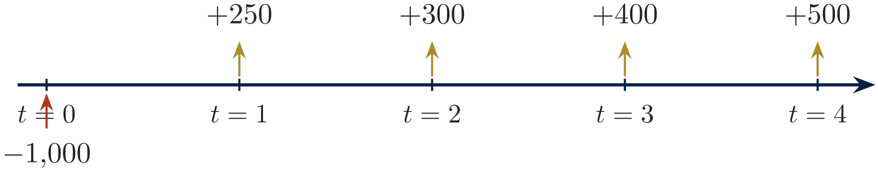
Move value through time

How can values at different dates be compared?

Build a timeline, compound forward, and discount backward to one common date.



A timeline is the grammar of valuation



- The sign records the decision maker's perspective, not whether a cash flow is "good."
- The spacing defines the period used by the interest rate.
- The timeline reveals whether a formula describes the actual contract.

Rule 1: compare or add values only at a common date

- "£900 + £1,000" is not a valuation when the amounts occur at different dates.
- Move both to today, both to year 2, or any other declared common date.
- Consistent market rates preserve the ranking whichever common date is chosen.

Interest is an exchange rate between dates

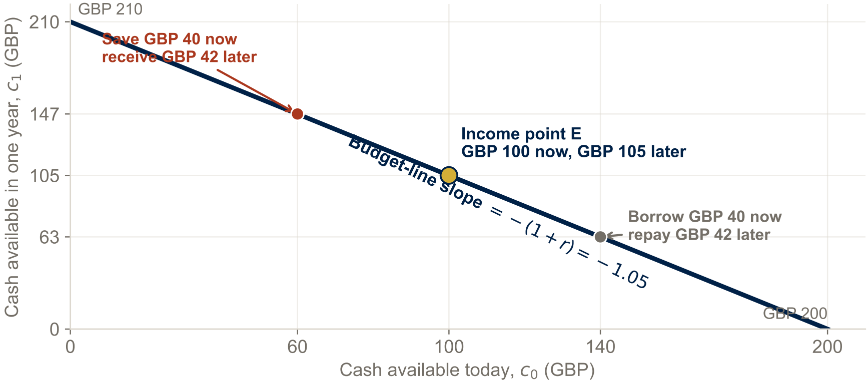
If the one-year annual effective rate is r, the market exchange is

$$£1 \text{ today} \quad \longleftrightarrow \quad £(1+r) \text{ in one year.}$$

r: one-year annual effective rate · $1+r$: gross accumulation factor

- Lending moves purchasing power forward: give up one pound now and receive $1+r$ pounds later.
- Borrowing reverses the exchange: receive now and promise $1+r$ pounds later.
- A rate is incomplete without its horizon, risk, currency, and quotation convention.

The same rate prices saving and borrowing across two dates



GBP = pound sterling · c_0: cash available today · c_1: cash available in one year · E: initial income point

At 5%, moving £40 across the two dates changes one-year resources by £42. The slope is $-(1+r)$: the interest rate is the relative price of dated cash.

Rule 2: compounding moves value forward

$$\mathrm{FV}_1 = C_0(1+r), \qquad \mathrm{FV}_T = C_0(1+r)^T.$$

FV_T: future value at date T · C_0: cash flow today · T: number of compounding periods

- Each period earns interest on the balance carried into that period.
- The exponent counts how many market exchanges separate today from the target date.
- The rate period must match the timeline period.

Appendix A2 · Derivation Python P1 · Time value

One-period future value: the interest earned is explicit

Deposit £8,000 for one year at a 4.5% annual effective rate.

$$\mathrm{FV}_1 = 8{,}000(1.045) = £8{,}360.$$

Opening balance	£8,000
Interest $= 0.045 \times 8{,}000$	£360
Closing balance	£8,360

For one period, simple and compound interest coincide; their difference appears only after interest itself begins to earn interest.

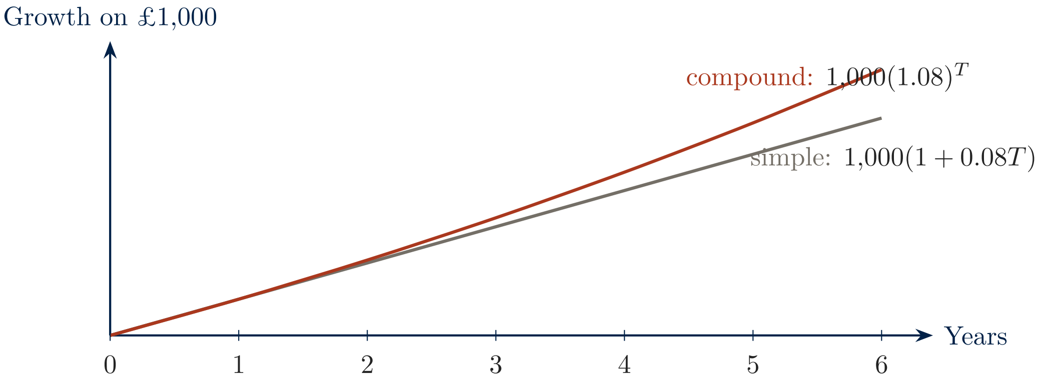
Multi-period growth is multiplicative

The same £8,000 remains invested for three years at 4.5%:

$$\mathrm{FV}_3 = 8{,}000(1.045)^3 = £9{,}129.33.$$

Year	Opening balance	Interest	Closing balance
1	8,000.00	360.00	8,360.00
2	8,360.00	376.20	8,736.20
3	8,736.20	393.13	9,129.33

Simple interest omits interest on interest



At 8% for six years, compounding adds £586.87; simple interest adds only £480.

The same equation can reveal an unknown rate or horizon

From $\mathrm{FV}_T = C_0(1+r)^T$,

$$r = \left(\frac{\mathrm{FV}_T}{C_0}\right)^{1/T} - 1, \qquad T = \frac{\ln(\mathrm{FV}_T / C_0)}{\ln(1+r)}.$$

- Doubling £1,000 to £2,000 in nine years requires $r = 2^{1/9} - 1 = 8.01\%$.
- At 6%, doubling takes $T = \ln 2 / \ln 1.06 = 11.90$ years.
- Logs convert repeated multiplication into a solvable linear relation in T.

Rule 3: discounting moves value backward

Compounding and discounting are inverse operations:

$$\mathrm{FV}_T = C_0(1+r)^T \qquad \Longleftrightarrow \qquad \mathrm{PV}_0 = \frac{C_T}{(1+r)^T}.$$

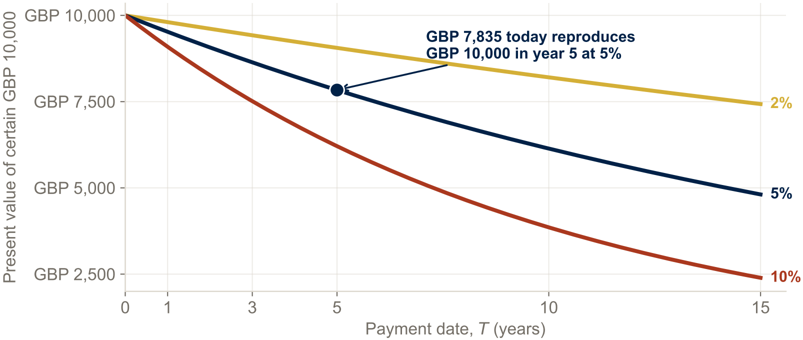
C_T: payment at date T · PV_0: equivalent value at date 0

- Discounting asks how much invested today would reproduce the future payment.
- A higher opportunity return means less money is needed today.
- A later payment is discounted through more periods.

Appendix A4 · Inverse operation and date invariance

One future payment: ask for the replicating amount today

$$\mathrm{PV}_0 = \frac{10{,}000}{(1.05)^5} = £7{,}835.26.$$



A later payment or a higher comparable return lowers present value because less money is needed today to reproduce the same dated payoff.

An uneven stream is valued one date at a time

For cash flows $C_0, \ldots, C_T$ under a flat effective rate r,

$$\mathrm{PV}_0 = \sum_{t=0}^{T} \frac{C_t}{(1+r)^t}.$$

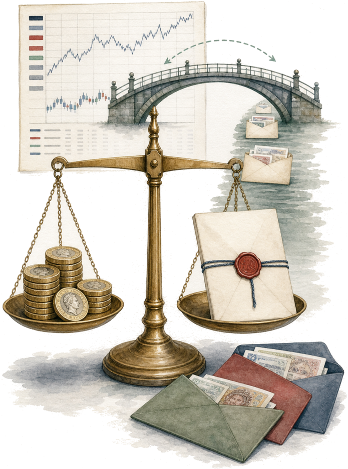
C_t is signed; the $t = 0$ term is not discounted

t	C_t	$1/(1.06)^t$	Discounted value
0	−1,000	1.0000	−1,000.00
1	300	0.9434	283.02
2	450	0.8900	400.50
3	500	0.8396	419.81
		Total	103.33

Present value and NPV

Why is present value a defensible comparison?

Use replication, opportunity cost, and the NPV rule to connect a calculation to an available market alternative.



A discount factor is the price of one future pound

$$d_t = \mathrm{PV}_0(£1 \text{ certain at } t), \qquad \mathrm{PV}_0(C_t) = d_t C_t.$$

d_t: date-0 discount factor for maturity t

- Under a flat annual effective rate, $d_t = (1 + r)^{-t}$.
- The discount factor is a price; the yield is a convention for quoting that price.
- Value additivity lets a multi-payment contract be treated as a portfolio of dated claims.

The Law of One Price disciplines valuation

If two traded portfolios deliver identical cash flows in every relevant date and state,

$$\boldsymbol{C}^A = \boldsymbol{C}^B \qquad \Longrightarrow \qquad P(A) = P(B)$$

$\boldsymbol{C}^A, \boldsymbol{C}^B$: complete dated, state-contingent cash-flow vectors · $P(A), P(B)$: current market prices

in an arbitrage-free market.

- Otherwise, buy the cheaper payoff and sell the more expensive copy.
- "Same expected payoff" is not enough; the dated state-by-state payoff must match.
- Frictions limit real trades but do not remove the benchmark logic.

Replication derives the price of a certain claim

A one-year deposit paying 4% can reproduce a certain £1,000 invoice:

$$P^* = \frac{1{,}000}{1.04} = \pounds 961.54.$$

P^*: no-arbitrage replicating price under the stated borrowing and lending opportunity

deposit £961.54 **receive £1,000**

today year 1

Discounting is the numerical expression of a replicating trade when borrowing and lending span the future payment.

If the claim is underpriced, the future cash flows finance the trade

Suppose the certain £1,000 claim trades at £950 while borrowing for one year costs 4%.

$+950 - 950 = 0$ $\qquad$ $+1{,}000 - 988 = +12$

today $\longrightarrow$ year 1

- Borrow £950 today and use it to buy the claim.
- In one year the claim pays £1,000; the loan requires £988.
- Net cash flow: zero today and £12 in one year.

The detailed trade table checks every date rather than reporting only the final profit.

Appendix A5 · Full arbitrage table

If the claim is overpriced, reverse the positions

Suppose the same claim trades at £975.

$+975 - 975 = 0$ $\qquad$ $+1{,}014 - 1{,}000 = +14$

today → year 1

- Short or sell the claim for £975 today.
- Lend the proceeds at 4%; the deposit becomes £1,014.
- Pay the promised £1,000 and retain £14 in one year.

Short-sale constraints, bid-ask spreads, taxes, and funding limits can prevent the literal trade; the pricing comparison remains informative.

The opportunity cost of capital is a market alternative

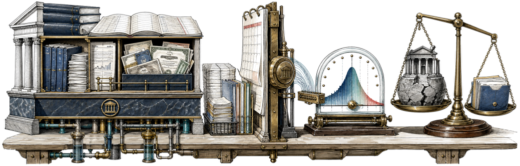
- It is the return available on an alternative with comparable timing and risk.
- It is not chosen because it is the firm's customary hurdle rate or the manager's target.
- Certain contractual payments can begin with near-risk-free market prices.
- Risky operating cash flows require a risk-adjusted comparison; equal expected cash flows need not have equal values.

> Day 1 uses certainty to establish the valuation engine. Day 4's CAPM material will address the required return on systematic risk.

CAPM = capital asset pricing model; it links required expected return to systematic market risk under its assumptions.

Project valuation follows four explicit steps

1. **Forecast** incremental cash flows, including timing and sign.
2. **Identify** a market opportunity cost matching maturity and risk.
3. **Discount** every cash flow to one comparison date.
4. **Decide** by comparing present value of benefits with present value of costs.



This sequence follows the corporate-finance textbooks but makes the market comparison and computational representation explicit.

Worked project: an energy retrofit changes operating cash flows

A warehouse retrofit costs £420,000 today and is forecast to save energy costs at each year-end:

Year	Saving	Discount factor at 7%	Present value
1	120,000	0.9346	112,150
2	140,000	0.8734	122,281
3	160,000	0.8163	130,608
4	100,000	0.7629	76,290
	520,000		441,328

Constructed case; savings are treated as certain nominal year-end cash flows for the calculation

NPV measures value created relative to the alternative

$$\text{NPV}_0 = \text{PV}_0(\text{benefits}) - \text{PV}_0(\text{costs}).$$

For the retrofit,

$$\text{NPV}_0 = 441{,}328 - 420{,}000 = £21{,}328.$$

- Accept an independent project when NPV is positive.
- Positive NPV means the project buys its dated payoff more cheaply than the comparison asset.
- The numerical result remains conditional on forecast cash flows and the 7% opportunity cost.

Mutually exclusive alternatives require incremental comparison

Suppose two retrofit designs cannot both be installed:

Design	PV benefits	Cost today	NPV
Standard	441,328	420,000	21,328
Advanced	520,000	485,000	35,000

- Both pass the independent accept/reject test.
- If capacity permits only one, choose the advanced design because its NPV is larger.
- Comparing IRRs or payback periods can rank mutually exclusive scale and timing patterns incorrectly.

IRR = internal rate of return, a discount rate that sets NPV to zero; payback period = time until cumulative undiscounted inflows recover the initial outlay.

Value additivity turns a contract into computable pieces

$$\mathrm{PV}_0(A+B)=\mathrm{PV}_0(A)+\mathrm{PV}_0(B), \qquad \mathrm{PV}_0=\boldsymbol{d}^\top\boldsymbol{C}.$$

$\boldsymbol{C}$: aligned vector of dated cash flows · $\boldsymbol{d}$: matching discount-factor vector

- Additivity explains why each payment can be priced separately and summed.
- The dot product is the computational form of the same economic principle.
- A dimension mismatch or date misalignment is a contract error, even if software returns a number.

Appendix A8 · Vector checks Python P2 · NPV arrays

IRR is a break-even rate; NPV remains the decision rule

For a cash-flow stream, an internal rate of return r^* solves

$$0 = \sum_{t=0}^{T} \frac{C_t}{(1+r^*)^t}.$$

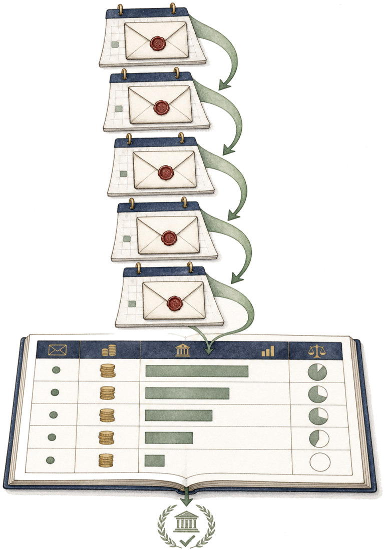
- For one initial outflow followed by positive inflows, IRR often gives a useful return summary.
- Accepting when r^* exceeds the opportunity cost is equivalent to positive NPV only under standard cash-flow patterns.
- Multiple sign changes can produce multiple IRRs; mutually exclusive projects can be ranked incorrectly.

IRR = internal rate of return; r^* is a root that makes NPV zero. Later we use the same root-finding idea for a financing threshold without treating IRR as a universal decision rule.

Repeated cash flows and loans

When can a long cash-flow list be compressed safely?

Recognise the timeline first; only then use perpetuities, annuities, growth formulas, and loan identities.

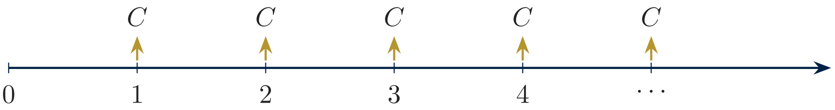


Repeated cash flows have recognisable shapes

Pattern	Timeline	Typical application
Perpetuity	level payments forever	endowment or preferred share
Ordinary annuity	level payments at $1, \ldots, T$	loan or lease payments
Annuity due	level payments at $0, \ldots, T-1$	rent paid in advance
Growing stream	payments grow at constant g	indexed cost or contribution plan

A shortcut is valid only when payment amount, spacing, first date, horizon, and rate period match its cash-flow pattern.

A perpetuity pays a level amount forever



$$\mathrm{PV}_0 = \frac{C}{r}.$$

First payment at $t = 1$; level payment C; constant effective rate $r > 0$

- A £80,000 annual endowment distribution requires £2.0m at 4%.
- The fund earns £80,000 each year and distributes the interest while preserving principal.

A delayed perpetuity is valued one date before it begins

A £50,000 annual payment begins at the end of year 4; $r = 5\%$.

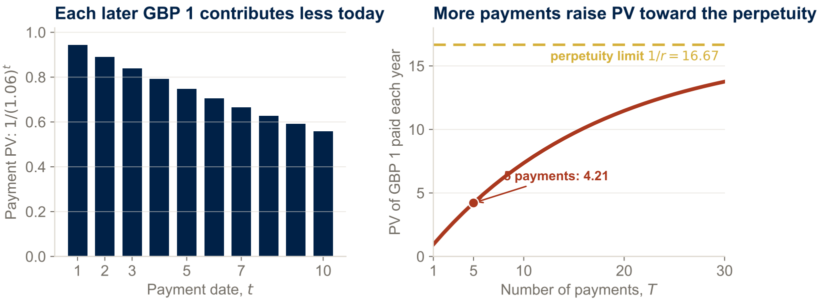
$$\mathrm{PV}_3 = \frac{50{,}000}{0.05} = 1{,}000{,}000, \qquad \mathrm{PV}_0 = \frac{1{,}000{,}000}{(1.05)^3} = \pounds 863{,}838.$$

- The perpetuity formula gives value at year 3, one period before the first payment.
- Discount that year-3 value through three periods to date 0.
- Discounting four periods would count the delay twice.

An ordinary annuity is a finite portfolio of dated claims

A level payment C arrives at dates $1, \ldots, T$:

$$\mathrm{PV}_0 = C \sum_{t=1}^{T} (1+r)^{-t}.$$



At 6%, later payments carry smaller present-value weights. Extending the horizon raises annuity value toward, but never above, the perpetuity limit C/r.

The annuity factor compresses the finite sum

$$\mathrm{PV}_0 = C\frac{1-(1+r)^{-T}}{r} = C \times \mathrm{PVAF}(r, T).$$

PVAF = present-value annuity factor · C: level end-of-period payment · T: number of payments · r: effective rate per payment period

- The first payment is one period from the valuation date.
- Payment frequency and rate frequency must agree.
- At $r = 6\%$, five annual payments of £2.15m have PV £9.0566m.

An annuity equals one perpetuity minus its unwanted tail

$$\underbrace{\frac{C}{r}}_{\text{payments } 1,2,\ldots} - \underbrace{\frac{C}{r(1+r)^T}}_{\text{payments } T+1, T+2,\ldots} = C\,\frac{1-(1+r)^{-T}}{r}.$$

- The first perpetuity supplies all desired annuity payments and an unwanted tail.
- The delayed perpetuity removes precisely the payments after date T.
- As $T \to \infty$, the tail's present value vanishes and the annuity approaches C/r.

Moving every payment one period earlier creates an annuity due

$$\mathrm{PV}_0^{\text{due}} = (1+r)\,\mathrm{PV}_0^{\text{ordinary}} = (1+r)C\,\frac{1-(1+r)^{-T}}{r}.$$

Pattern	Payment dates	Value relation
Ordinary annuity	$1, \ldots, T$	base formula
Annuity due	$0, \ldots, T-1$	ordinary value $\times(1+r)$

Future value accumulates each contribution to the target date

For T end-of-period contributions,

$$\mathrm{FV}_T = C\frac{(1+r)^T - 1}{r}.$$

- The payment at $t = 1$ compounds for $T - 1$ periods.
- The final payment at $t = T$ earns no interest before the target date.
- Investing £5,000 annually for ten years at 5% produces £62,889.46.

The future-value factor is the present-value annuity factor carried forward by $(1+r)^T$.

A growing perpetuity separates discounting from cash-flow growth

If the first payment at $t = 1$ is C_1 and payments grow at rate g,

$$\mathrm{PV}_0 = \frac{C_1}{r - g}, \qquad r > g.$$

$C_t = C_1(1+g)^{t-1}$ · r and g use the same period and basis

- A 2% indexed £80,000 first distribution requires £4.0m when $r = 4\%$.
- Growth reduces the net rate at which distant payments lose present value.
- When $g \geq r$, the infinite discounted sum does not converge under these assumptions.

A growing annuity has a finite horizon

For payments $C_1, C_1(1+g), \ldots, C_1(1+g)^{T-1}$,

$$\mathrm{PV}_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right], \qquad r \neq g.$$

- Use this for a finite indexed lease, contribution plan, or cost stream.
- With $C_1 = £10,000$, $g = 3\%$, $r = 7\%$, $T = 8$, PV is £65,682.
- At $r = g$, each discounted payment equals $C_1/(1+r)$; the limiting value is $TC_1/(1+r)$.

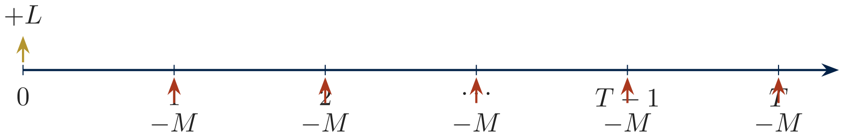
Appendix A14 · Full derivation and $r = g$

A loan exchanges principal today for payments later

Borrow L today and repay M at the end of each of T periods:

$$L = M\frac{1-(1+r)^{-T}}{r}.$$

L: loan principal received at date 0 · M: equal end-of-period payment · T: number of payments



Solve for the equal payment, not by trial and error

$$M = L\frac{r}{1-(1+r)^{-T}}.$$

Borrow £20,000 for four years at 6% with annual payments:

$$M = 20{,}000\frac{0.06}{1-(1.06)^{-4}} = £5{,}771.83.$$

- Total paid is £23,087.32, but payments occur at different dates.
- Their present value at the contractual rate is exactly the £20,000 principal.

Appendix A15 · Payment algebra   Python P3 · Loan schedule

Each payment contains interest and principal repayment

Let B_{t-1} be the opening balance before payment t:

$$\text{Interest}_t = rB_{t-1}, \qquad \text{Principal}_t = M - rB_{t-1}, \qquad B_t = B_{t-1}(1+r) - M.$$

Year	Opening	Interest	Principal	Closing
1	20,000.00	1,200.00	4,571.83	15,428.17
2	15,428.17	925.69	4,846.14	10,582.03

The outstanding balance is the PV of remaining payments

After payment k, with $T - k$ payments left,

$$B_k = M \frac{1 - (1+r)^{-(T-k)}}{r}.$$

- The backward-looking recursion and forward-looking PV must produce the same balance.
- This identity is a powerful check on an amortisation schedule.
- Settlement fees or prepayment penalties are additional dated cash flows, not part of the annuity identity.

Non-annual cash flows require period alignment

For a 6% APR compounded monthly, the monthly effective rate is

$$r_m = \frac{0.06}{12} = 0.5\%.$$

A five-year monthly loan therefore uses $T = 60$, not $T = 5$:

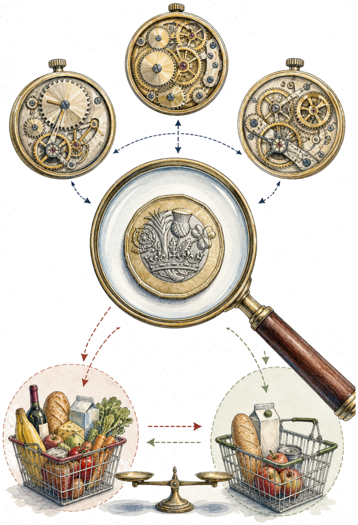
$$M = L\frac{0.005}{1 - (1.005)^{-60}}.$$

- Cash-flow period: one month.
- Rate period: one month.
- Number of periods: 60 months.

Rate language and purchasing power

When do two rate quotations give the same price?

Align compounding and payment periods, incorporate fees, and distinguish nominal from real returns.



A usable rate carries a passport

Passport field	What must match the cash flow?
Maturity	overnight, one month, one year, or a full term structure
Convention	EAR, APR plus frequency, simple, or continuous
Currency	sterling, dollars, euros, or another unit
Inflation basis	nominal pounds or constant-purchasing-power pounds
Risk and contract	certainty, default, collateral, options, liquidity
Fees and tax	all-in dated cash flows, not only the headline percentage

EAR measures the actual one-year growth factor

$$£1 \text{ today} \longrightarrow £(1 + r_{\text{EAR}}) \text{ after one year.}$$

£1 one realised annual growth factor **£1.126825**

start of year → end of year

- EAR includes the effect of compounding within the year.
- It can be used directly in a one-year accumulation or discount factor.
- A 12.6825% EAR means £1 becomes £1.126825 after one year.

EAR = effective annual rate. It describes a one-year price but does not by itself say whether the cash flow is risk-free, nominal, or in sterling.

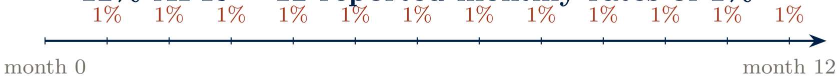
APR annualises a periodic rate without intra-year compounding

If the periodic effective rate is r_p and there are m periods per year,

$$q_m = mr_p.$$

APR = annual percentage rate · q_m: nominal APR compounded m times per year · r_p: effective rate per subperiod

12% APR = 12 reported monthly rates of 1%



- A 12% APR compounded monthly reports a 1% monthly effective rate.
- The annual growth factor is $(1.01)^{12}$, not 1.12.
- The compounding frequency is part of the quote and cannot be omitted.

Worked conversion: 12% APR monthly is 12.6825% EAR

$$r_m = \frac{0.12}{12} = 0.01, \qquad r_{\text{EAR}} = (1.01)^{12} - 1 = 0.126825.$$

- Monthly cash flow: use 1% per month.
- One-year cash flow: use 12.6825% EAR.
- Both quotations imply the same one-year discount factor $1/(1.01)^{12}$.

Payment frequency and rate frequency must be identical

A 9% EAR does not imply a 9%/4 quarterly rate. The equivalent quarterly rate is

$$r_q = (1.09)^{1/4} - 1 = 2.1778\%.$$

- Four quarters at 2.1778% compound to exactly 9% over the year.
- Using 2.25% per quarter would imply a 9.3083% EAR.
- Convert the rate before valuing quarterly payments.

Continuous compounding is another quotation language

A continuously compounded annual rate z implies

$$\mathrm{FV}_T = C_0 e^{zT}, \qquad \mathrm{PV}_0 = C_T e^{-zT}.$$

- The economic operation is unchanged: move value between dates.
- A 4.20% continuous rate has EAR $e^{0.042} - 1 = 4.2894\%$.
- Market curve files often use continuous rates because maturity calculations become additive.

z: continuously compounded annual rate; e: base of the natural logarithm; T: horizon in years on the same day-count basis.

Equivalent quotations preserve the same price

$$1 + r_{\text{EAR}} = \left(1 + \frac{q_m}{m}\right)^m = e^z.$$

Quote	One-year growth factor	Use in a matching period
EAR	$1 + r_{\text{EAR}}$	annual cash flow
APR q_m	$(1 + q_m/m)^m$	q_m/m each subperiod
Continuous z	e^z	e^{-zT} discount factor

Fees belong in dated net cash flows

A borrower receives £9,800 after a £200 fee and repays £850 monthly for 12 months. The all-in monthly rate r_m solves

$$9{,}800 = 850\frac{1-(1+r_m)^{-12}}{r_m}.$$

Numerically, $r_m = 0.6209\%$, so

$$\text{EAR} = (1+r_m)^{12} - 1 = 7.71\%.$$

The advertised rate is incomplete when mandatory fees change the cash actually received or repaid.

Nominal and real returns answer different questions

- **Nominal return:** growth in pounds without removing inflation.
- **Real return:** growth in the quantity of goods and services those pounds can buy.
- Match nominal cash flows with nominal rates and real cash flows with real rates.

Mixing a real cash-flow forecast with a nominal discount rate double-counts inflation and understates value.

The Fisher relation separates money growth from price growth

$$1 + r_n = (1 + r_r)(1 + \pi), \qquad r_r = \frac{1 + r_n}{1 + \pi} - 1.$$

r_n: nominal return · r_r: real return · π: inflation over the same horizon

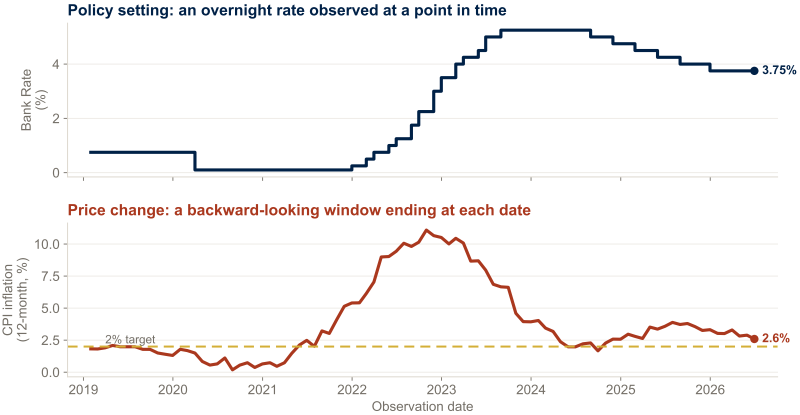
If $r_n = 6\%$ and inflation is 3%,

$$r_r = \frac{1.06}{1.03} - 1 = 2.9126\%.$$

The approximation $r_r \approx r_n - \pi = 3\%$ is close here, but the exact relation preserves compound growth.

Appendix A19 · Derive the exact relation

A policy rate and inflation are not horizon-matched inputs



CPI = Consumer Prices Index · Bank Rate = Bank of England's overnight policy rate

Bank of England Bank Rate series IUDBEDR; ONS CPI 12-month inflation D7BT; frozen 13 August 2026.

Do not subtract two nearby observations mechanically

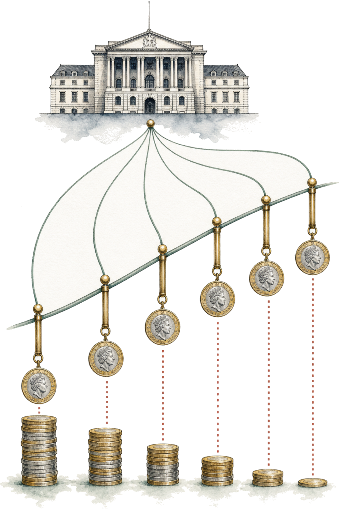
- Bank Rate is an overnight policy rate at a point in time.
- CPI inflation is a backward-looking 12-month change in a consumer price index.
- A forward-looking real return requires a nominal rate and expected inflation over the same horizon.
- The chart is evidence about changing monetary conditions, not a ready-made project discount rate.

The next section replaces one short policy rate with maturity-specific market prices.

From one rate to a discount curve

Which market rate prices each payment date?

Treat the flat rate as a special case, read the term structure, and attach one discount factor to each maturity.



A flat rate is a simplifying assumption, not a market law

Under one constant annual effective rate,

$$d_t = (1 + r)^{-t}.$$

Under a term structure,

$$d_t = (1 + y_t)^{-t},$$

where y_t can differ by maturity.

y_t: annual effective spot yield for maturity t; the collection of maturity-specific yields is the term structure

- A five-payment contract contains five dated claims.
- Market prices generally imply a different discount factor for each date.
- The flat-rate formula remains useful for quoting and sensitivity analysis.

Discount factors are prices; spot yields quote those prices

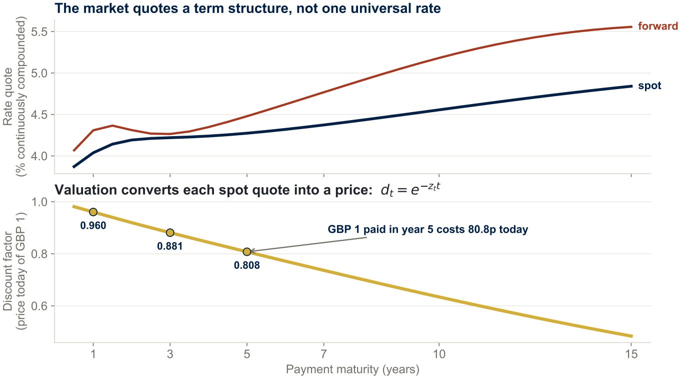
Under annual compounding,

$$d_t = \frac{1}{(1+y_t)^t}, \qquad y_t = d_t^{-1/t} - 1.$$

d_t: price today of certain £1 at t · y_t: t-year annual effective spot yield

- Change the quotation convention and the yield changes.
- Preserve the discount factor and the cash-flow price does not change.

A curve supplies a price for every payment date



OIS = overnight index swap; sterling OIS references SONIA, the Sterling Overnight Index Average. A spot rate prices date 0 to maturity t; an instantaneous forward rate prices a local future interval implied by today's curve.

Bank of England fitted sterling OIS spot and instantaneous forward curves, 12 August 2026; continuously compounded.

Translate each curve point into a cash-flow price

For a continuously compounded spot yield z_t,

$$d_t = e^{-z_t t}.$$

Maturity	Spot z_t	Discount factor	PV of certain £1m
1 year	4.038%	0.96042	£0.96042m
3 years	4.220%	0.88108	£0.88108m
5 years	4.274%	0.80759	£0.80759m

A five-year yield is not inserted into a one-year cash-flow denominator; each maturity supplies its own price.

Curve-based valuation is a date-aligned dot product

$$\mathrm{PV}_0 = \sum_{t=1}^{T} C_t e^{-z_t t} = \boldsymbol{d}^\top \boldsymbol{C}.$$

- Year-1 cash flow uses z_1; year-5 cash flow uses z_5.
- The dot product exposes the computational structure: dates, cash flows, and discount factors must align.
- A flat rate is recovered when every z_t is generated by the same annual growth rule.

Appendix A23 · OIS data Python P5 · Curve valuation

Worked stream: one price per maturity

Consider certain year-end receipts of £2m, £3m, and £4m.

Year	Cash flow	Spot z_t	$e^{-z_t t}$	PV (£m)
1	2.000	4.038%	0.96042	1.92084
2	3.000	4.192%	0.91957	2.75871
3	4.000	4.220%	0.88108	3.52432
	9.000			8.20387

GBP = pound sterling; amounts are in millions; spot curve dated 12 August 2026

A flat rate and a curve weight maturities differently

Use the same continuous-compounding convention in both valuations:

$$V_{\text{flat}} = \pounds 8.2025\text{m} \qquad \text{and} \qquad V_{\text{curve}} = \pounds 8.2039\text{m}.$$

- The values are close because the selected flat rate lies near the relevant curve points.
- Using the same quotation basis isolates the effect of the curve's shape and cash-flow timing.
- Here the curve effect is only about £1,400; back-loaded contracts can produce a much larger gap.

A flat rate can approximate a curve for a particular contract; it is not a replacement for the term structure in every contract.

A forward rate equates two investment paths

Under continuous compounding, invest to b directly or invest to a and then from a to b:

$$e^{z_b b} = e^{z_a a} e^{f_{a,b}(b-a)}, \qquad f_{a,b} = \frac{z_b b - z_a a}{b - a}.$$

From the curve, $z_1 = 4.038\%$ and $z_2 = 4.192\%$, so

$$f_{1,2} = 2z_2 - z_1 = 4.347\%.$$

$f_{a,b}$: continuously compounded forward rate fixed today for the future interval from date a to date b

The forward rate is a break-even rate, not a guaranteed forecast

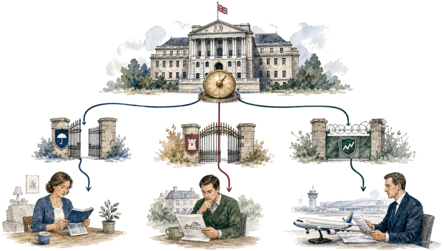
The no-arbitrage break-even condition is

$$e^{z_b b} = e^{z_a a} e^{f_{a,b}(b-a)}.$$

- It is implied today by relative prices at two maturities.
- It is the future reinvestment rate that makes the two strategies equal before frictions.
- Expectations can influence the curve, but so can risk premia, liquidity, collateral, and market segmentation.
- A forward rate therefore need not equal the future realised short rate.

Break-even means equal terminal wealth for the two declared paths. Use the forward curve to interpret today's prices; label any forecast interpretation as an additional assumption.

Bank Rate, OIS, and corporate borrowing price different objects



- Bank Rate prices an overnight policy instrument, not a five-year payment stream.
- SONIA-based OIS supplies collateralised near-risk-free term benchmarks, not a corporate loan.
- Loan quotes and corporate yields add contract, credit, liquidity, collateral, fee, and option terms.

The benchmark is a starting price for time; the decision requires the firm's executable alternative.

A benchmark curve requires explicit adjustments

1. Start with maturity-specific prices in the correct currency and nominal/real basis.
2. Match the decision maker's actual borrowing, lending, or investment alternative.
3. Put fees, tax, collateral, and prepayment terms into dated cash flows where possible.
4. Represent uncertain payment amounts consistently; do not count expected loss in both cash flows and a full spread.
5. Report sensitivity when the appropriate adjustment cannot be observed precisely.

$$V_0 = \sum_{t=1}^{T} C_t^{\text{contract}} d_t^{\text{opportunity}}.$$

C_t^{contract}: executable net contractual cash flow · $d_t^{\text{opportunity}}$: date- and risk-matched price from the decision maker's available alternative

Checkpoint: identify the rate mismatch before calculating

Mismatch	First repair
Monthly lease uses 8% as a monthly rate	convert the quote to a monthly effective rate
Nominal invoice uses a real discount rate	use a nominal rate on the same inflation basis
Five-year project uses today's Bank Rate	use maturity-specific market prices
Risky receivable is treated as certain	model states or use a consistent risk adjustment

Repair sequence: align period, inflation basis, maturity, and risk before entering a rate in the calculation.

Integrated decision

Can valuation support a sound recommendation?

Represent a financing quote, value it under alternative curves, find decision thresholds, and state what remains unknown.

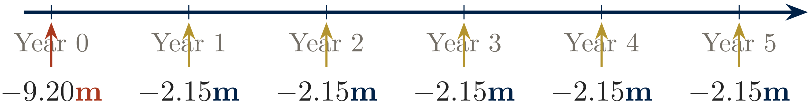
One financing quote brings the pieces together

A UK manufacturer plans to acquire production equipment. The supplier offers:

- **Cash plan:** pay £9.20m today.
- **Instalment plan:** pay £2.15m at each of the next five year-ends.
- **Quoted financing rate:** 6%, stated to be an annual effective rate.

> This is a constructed financing case. It compares rate assumptions transparently without attempting to reproduce every contractual feature of a real procurement agreement.

Write both plans on one timeline before choosing a formula



- Both are cost legs from the manufacturer's perspective.
- The undiscounted instalment total is £10.75m, but it cannot be compared directly with £9.20m today.
- All amounts are fixed nominal pounds; payment dates are year-ends.

The computational representation must preserve the contract

$$\boldsymbol{t} = (1, 2, 3, 4, 5), \qquad \boldsymbol{P} = (2.15, 2.15, 2.15, 2.15, 2.15).$$

$$\boldsymbol{d}(r) = \left((1+r)^{-1}, \ldots, (1+r)^{-5}\right), \qquad V(r) = \boldsymbol{d}(r)^{\top}\boldsymbol{P}.$$

Amounts in GBP million; $P_t > 0$ records payment-cost magnitudes

- Check equal vector lengths and increasing dates.
- Check $d_0 = 1$, $0 < d_t \leq 1$, and decreasing discount factors for positive rates.
- Reconcile the compact annuity formula with the explicit dot product.

At a flat 6% EAR, the instalments have lower present value

$$V(6\%) = 2.15\frac{1-(1.06)^{-5}}{0.06} = £9.0566\text{m}.$$

- Cash plan: £9.2000m today.
- Instalment advantage: $9.2000 - 9.0566 = £0.1434$m.
- Conditional recommendation: choose instalments if 6% is the relevant effective annual opportunity cost for all five dates.

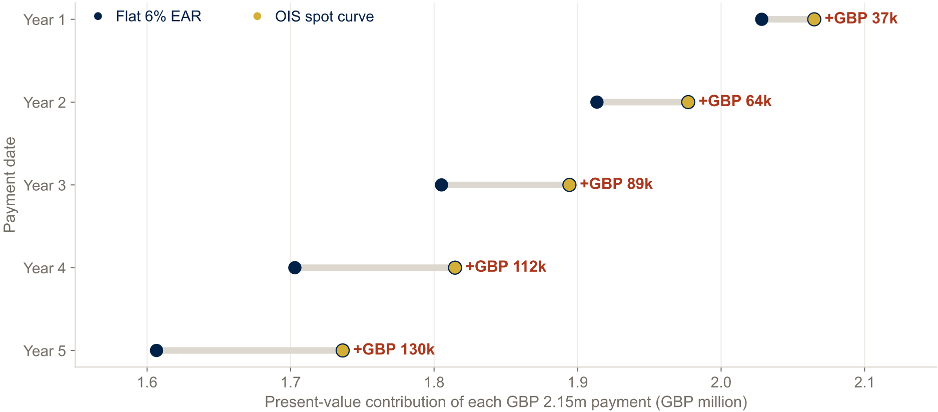
The observed OIS curve reverses the benchmark comparison

$$V_{\mathrm{OIS}} = 2.15 \sum_{t=1}^{5} e^{-z_t t} = £9.4873\text{m}.$$

- Instalments exceed the cash price by £0.2873m in present value.
- Conditional OIS-benchmark recommendation: choose cash.
- The reversal demonstrates that rate selection is economically material.

OIS is a transparent benchmark, not automatically the manufacturer's final contract-specific opportunity cost.

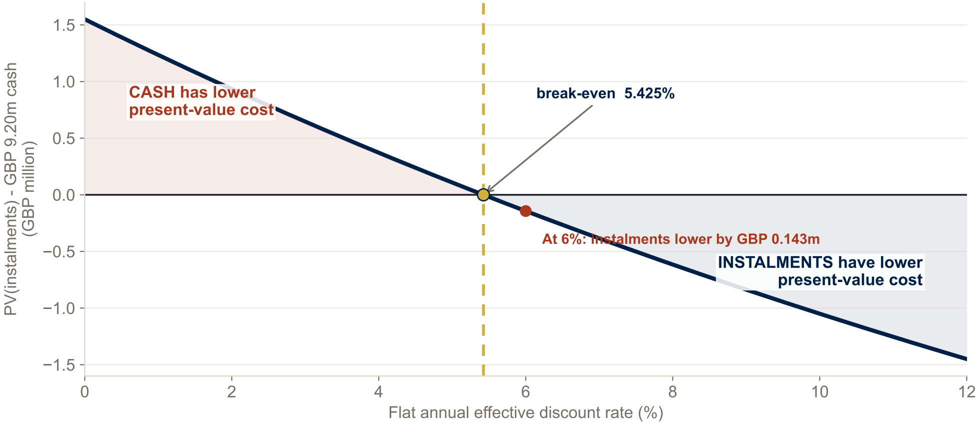
Lower OIS spot rates raise every instalment's present value



The five payment-level increases sum to £0.431m: enough to reverse the flat-6% comparison.

Each point is one £2.15m payment times its date-matched discount factor; OIS curve dated 12 August 2026.

The flat-rate decision changes at 5.425%



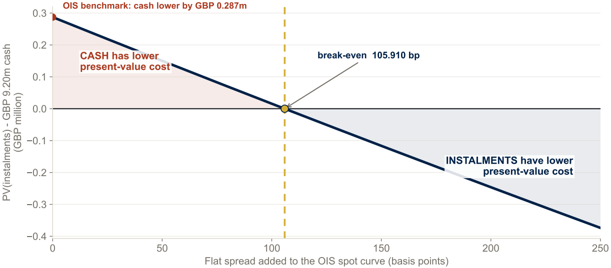
The intersection solves $2.15 \sum_{t=1}^{5} (1 + r^*)^{-t} = 9.20$.

Appendix A26 · Root finding Python P6 · Decision root

A spread threshold creates a decision boundary

$$\Delta(s) = \sum_{t=1}^{5} 2.15e^{-(z_t+s)t} - 9.20, \qquad \Delta(s^*) = 0 \Longrightarrow s^* = 105.9 \text{ bp}.$$

s: continuously compounded spread added to every OIS spot rate · $\Delta(s)$: instalment PV minus the cash price



bp = basis point = 0.01 percentage point. A threshold reports how much contract-specific funding cost is needed to overturn the OIS-benchmark recommendation.

A decision memo separates calculation from judgement

1. **Contract:** cash £9.20m today versus five nominal year-end payments of £2.15m.
2. **Results:** instalment PV is £9.0566m at flat 6%; £9.4873m under OIS.
3. **Thresholds:** 5.425% flat EAR or 105.9 bp over OIS.
4. **Missing information:** all-in funding cash flows, fees, tax, collateral, prepayment rights, and payment certainty.
5. **Recommendation:** obtain the firm's executable all-in financing alternative before committing.

Day 1 leaves one reusable valuation engine

1. Draw dated, signed cash flows.
2. Choose one comparison date.
3. Translate rate quotations into matching discount factors.
4. Multiply each cash flow by its date-specific price and add.
5. Compare alternatives by NPV and report assumptions and thresholds.

> Day 2 applies exactly this engine to bonds: coupons are an annuity-like stream, principal is a final payment, and market yields determine their price and interest-rate exposure.

Technical appendix map

← **Return to the Day 1 synthesis**

Practical map

Appendix A1: Timeline and sign audit

1. Declare the decision maker's perspective.
2. List exact payment dates before replacing them with period numbers.
3. Record receipts as positive and payments as negative.
4. Confirm that one rate period equals one timeline interval.
5. Mark uncertain, indexed, or optional cash flows separately.

← **Return to the timeline**

Appendix map

Appendix A2: Deriving compound growth

$$\begin{aligned} C_1 &= C_0(1+r), \\ C_2 &= C_1(1+r) = C_0(1+r)^2, \\ C_3 &= C_2(1+r) = C_0(1+r)^3. \end{aligned}$$

Repeating the same substitution for T periods gives

$$C_T = C_0(1+r)^T.$$

The exponent counts repeated multiplication by the gross return, not calendar years unless the rate period is annual.

← **Return to compounding**

Appendix map

Appendix A3: Solving for an unknown rate or horizon

Starting from $\mathrm{FV}_T = C_0(1+r)^T$:

$$\begin{aligned} \frac{\mathrm{FV}_T}{C_0} &= (1+r)^T, \\ \left(\frac{\mathrm{FV}_T}{C_0}\right)^{1/T} &= 1+r, \\ r &= \left(\frac{\mathrm{FV}_T}{C_0}\right)^{1/T} - 1. \end{aligned}$$

Taking logs instead gives

$$T = \frac{\ln(\mathrm{FV}_T / C_0)}{\ln(1+r)}.$$

← **Return to unknown rate or horizon**

Appendix map

Appendix A4: Discounting and comparison-date invariance

From $C_T = C_0(1+r)^T$, divide by $(1+r)^T$:

$$C_0 = \frac{C_T}{(1+r)^T}.$$

If every dated cash flow is moved to date k at the same flat rate r, then for alternative j

$$V_k^j = \sum_t C_t^j (1+r)^{k-t} = (1+r)^k V_0^j.$$

Thus $V_0^A > V_0^B$ if and only if $V_k^A > V_k^B$. For $t < k$, this comparison assumes interim cash flows can be reinvested or financed at the same rate.

← **Discounting** ← **NPV**

Appendix map

Appendix A5: Underpriced claim — full arbitrage table

Trade	$t = 0$	$t = 1$
Borrow £950 at 4%	$+950$	-988
Buy certain £1,000 claim	-950	$+1{,}000$
Net cash flow	**0**	**+12**

The trade has no initial cost and a positive certain future payoff under the stated frictionless assumptions.

← **Return to underpricing**

Appendix map

Appendix A6: Overpriced claim — reverse-trade audit

Trade	$t = 0$	$t = 1$
Short/sell claim	+975	−1,000
Lend £975 at 4%	−975	+1,014
Net cash flow	**0**	**+14**

In a market with bid-ask spreads, use executable borrowing, lending, buying, and selling prices in the audit.

← **Return to overpricing**

Appendix map

Appendix A7: Equal expected payoff does not imply equal price

Claim	Good state ($p = 0.5$)	Bad state ($p = 0.5$)	Expected payoff
A: certain	100	100	100
B: state-dependent	200	0	100

Claim A pays in both states; Claim B pays only in the good state. Their prices depend on state prices and risk exposure, not only the arithmetic mean payoff.

← **Return to opportunity cost**

Appendix map

Appendix A8: Vector implementation and validation identities

$$\mathrm{PV} = \boldsymbol{d}^{\top}\boldsymbol{C} = \sum_{t=0}^{T} d_t C_t.$$

- Shape check: $\mathrm{len}(\boldsymbol{d}) = \mathrm{len}(\boldsymbol{C})$.
- Date check: discount factors correspond to the same ordered dates as cash flows.
- Unit check: rates are decimals; cash-flow units are consistent.
- Identity check: $d_0 = 1$; under positive certain rates, $0 < d_t \leq 1$.
- Formula check: explicit cash-flow sum equals any valid annuity shortcut.

$\mathrm{len}(\cdot)$: number of entries in a vector; $d_0 = 1$ because a date-0 pound is already at the valuation date.

← **Cash-flow sum** ← **Additivity** ← **Curve valuation**

Appendix map

Appendix A9: Perpetuity geometric-series derivation

Let $x = (1+r)^{-1}$, so $0 < x < 1$ when $r > 0$:

$$\begin{aligned}
\mathrm{PV}_0 &= Cx + Cx^2 + Cx^3 + \cdots\,, \\
x\,\mathrm{PV}_0 &= Cx^2 + Cx^3 + Cx^4 + \cdots\,, \\
(1-x)\,\mathrm{PV}_0 &= Cx, \\
\mathrm{PV}_0 &= \frac{Cx}{1-x} = \frac{C}{r}.
\end{aligned}$$

← **Return to perpetuities**

Appendix map

Appendix A10: Delayed perpetuity timing

If the first payment C occurs at date $k+1$, the perpetuity is worth C/r at date k:

$$\mathrm{PV}_k = \frac{C}{r}.$$

Move that value from date k to date 0:

$$\mathrm{PV}_0 = \frac{C/r}{(1+r)^k}.$$

The exponent is k, not $k+1$, because C/r is already the value one period before the first payment.

← **Return to delayed perpetuities**

Appendix map

Appendix A11: Ordinary-annuity derivation

Let $x = (1+r)^{-1}$:

$$
\begin{aligned}
\mathrm{PV}_0 &= C(x + x^2 + \cdots + x^T), \\
x\,\mathrm{PV}_0 &= C(x^2 + x^3 + \cdots + x^{T+1}), \\
(1-x)\,\mathrm{PV}_0 &= C(x - x^{T+1}), \\
\mathrm{PV}_0 &= C\,\frac{x(1-x^T)}{1-x} = C\,\frac{1-(1+r)^{-T}}{r}.
\end{aligned}
$$

← **Return to the annuity factor**

Appendix map

Appendix A12: Annuity-due and future-value identities

Shifting each ordinary-annuity payment one period earlier gives

$$\mathrm{PV}_0^{\mathrm{due}} = (1+r)\,\mathrm{PV}_0^{\mathrm{ordinary}}\,.$$

Carrying the ordinary annuity's PV to date T gives

$$\begin{aligned}\mathrm{FV}_T &= (1+r)^T C\,\frac{1-(1+r)^{-T}}{r}\\ &= C\,\frac{(1+r)^T-1}{r}.\end{aligned}$$

← **Return to payment timing**

Appendix map

Appendix A13: Growing-perpetuity derivation and convergence

$$\begin{aligned}
\mathrm{PV}_0 &= \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \cdots \\
&= \frac{C_1}{1+r}\left[1 + q + q^2 + \cdots\right], \qquad q = \frac{1+g}{1+r}.
\end{aligned}$$

The series converges when $|q| < 1$; for ordinary positive rates this requires $r > g$. Then

$$\mathrm{PV}_0 = \frac{C_1}{1+r}\frac{1}{1-q} = \frac{C_1}{r-g}.$$

← **Return to growing perpetuities**

Appendix map

Appendix A14: Growing-annuity derivation and the $r = g$ case

With $q = (1+g)/(1+r)$,

$$\mathrm{PV}_0 = \frac{C_1}{1+r} \sum_{t=0}^{T-1} q^t = \frac{C_1}{1+r} \frac{1-q^T}{1-q}.$$

Simplifying for $r \neq g$ gives

$$\mathrm{PV}_0 = \frac{C_1}{r-g} \left[1 - \left(\frac{1+g}{1+r} \right)^T \right].$$

If $r = g$, then $q = 1$ and every discounted payment equals $C_1/(1+r)$, hence $\mathrm{PV}_0 = TC_1/(1+r)$.

← **Return to growing annuities**

Appendix map

Appendix A15: Solving the equal-payment loan

The principal equals the PV of payments:

$$L = M\frac{1-(1+r)^{-T}}{r}.$$

Divide both sides by the annuity factor:

$$M = \frac{L}{[1-(1+r)^{-T}]/r} = L\frac{r}{1-(1+r)^{-T}}.$$

Verification: substitute the calculated payment back into the PV equation; the result should recover L up to rounding.

← **Return to loan payments**

Appendix map

Appendix A16: Complete amortisation audit

Year	Opening	Payment	Interest	Principal	Closing
1	20,000.00	5,771.83	1,200.00	4,571.83	15,428.17
2	15,428.17	5,771.83	925.69	4,846.14	10,582.03
3	10,582.03	5,771.83	634.92	5,136.91	5,445.12
4	5,445.12	5,771.83	326.71	5,445.12	0.00

Rounding the displayed payment can leave a few pence in the final balance; production schedules retain full precision internally.

← **Return to amortisation**

Appendix map

Appendix A17: General non-annual period conversion

If an EAR is known and there are m equal periods per year,

$$r_p = (1 + r_{\mathrm{EAR}})^{1/m} - 1.$$

If a nominal APR q_m compounded m times is known,

$$r_p = \frac{q_m}{m}.$$

Over Y years, use $T = mY$ payment periods. For irregular dates, use exact year fractions and discount factors rather than forcing an annuity formula.

← **Return to period alignment**

Appendix map

Appendix A18: APR, EAR, and the continuous limit

APR = annual percentage rate · EAR = effective annual rate · m: compounding periods per year

For APR q_m compounded m times,

$$1 + r_{\text{EAR}} = \left(1 + \frac{q_m}{m}\right)^m .$$

Therefore

$$q_m = m\left[(1 + r_{\text{EAR}})^{1/m} - 1\right].$$

Holding a continuous quote z fixed and taking the compounding frequency to infinity,

$$\lim_{m\to\infty}\left(1 + \frac{z}{m}\right)^m = e^z, \qquad r_{\text{EAR}} = e^z - 1.$$

← **Return to APR/EAR conversion**

Appendix map

Appendix A19: Deriving the exact Fisher relation

Let P_0 be today's price level. One pound invested at nominal rate r_n becomes $1 + r_n$ pounds. If prices grow by $1 + \pi$, purchasing power becomes

$$\frac{1 + r_n}{1 + \pi}.$$

Define this purchasing-power growth as $1 + r_r$:

$$1 + r_r = \frac{1 + r_n}{1 + \pi} \quad \Longleftrightarrow \quad 1 + r_n = (1 + r_r)(1 + \pi).$$

← **Return to nominal and real rates**

Appendix map

Appendix A20: Fees and the all-in effective rate

Net proceeds at date 0 are £9,800; repayments are £850 for 12 months. Define

$$f(r_m) = 850\frac{1-(1+r_m)^{-12}}{r_m} - 9{,}800.$$

Because $f(0) = 400 > 0$ and $f(0.02) < 0$, a numerical root exists in that interval. Brent's method gives

$$r_m = 0.00620898, \qquad (1+r_m)^{12} - 1 = 7.7106\%.$$

Brent's method is a bracketed numerical root finder: it searches an interval whose endpoints give opposite signs.

← **Return to all-in borrowing cost**

Appendix map

Appendix A21: Discount-factor and yield conversions

Annual effective spot yield:

$$d_t = (1 + y_t)^{-t}, \qquad y_t = d_t^{-1/t} - 1.$$

Continuous spot yield:

$$d_t = e^{-z_t t}, \qquad z_t = -\frac{\ln d_t}{t}.$$

Both quote the same price when

$$1 + y_t = e^{z_t}.$$

← **Return to discount factors and yields**

Appendix map

Appendix A22: Forward rate from two equivalent paths

Path 1 invests one pound directly from 0 to b: $e^{z_b b}$. Path 2 invests from 0 to a, then locks the interval $[a, b]$:

$$e^{z_a a} e^{f_{a,b}(b-a)}.$$

No-arbitrage equates the terminal payoffs:

$$z_b b = z_a a + f_{a,b}(b-a),$$

$$f_{a,b} = \frac{z_b b - z_a a}{b-a}.$$

← **Return to forward rates**

Appendix map

Appendix A23: OIS data construction and provenance

- Source: Bank of England yield-curve archive, workbook *OIS daily data current month.xlsx*.
- Valuation date: 12 August 2026, the latest complete spot/forward observation when assets were frozen.
- Source sheets: fitted spot curve and instantaneous forward curve.
- Quotation: annual continuously compounded sterling rates.
- Transformation: $d_t = e^{-z_t t}$; annual maturities 1–5 selected for the financing case.
- OIS rates are model-based fitted estimates and are used as a transparent near-risk-free benchmark.

OIS = overnight index swap; SONIA = Sterling Overnight Index Average; z_t is the fitted continuously compounded spot rate for maturity t.

← **Return to the OIS curve**

Appendix map

Appendix A24: Flat 6% equipment-financing audit

Year	Payment	$1/(1.06)^t$	PV contribution
1	2.150	0.943396	2.028302
2	2.150	0.889996	1.913492
3	2.150	0.839619	1.805181
4	2.150	0.792094	1.703001
5	2.150	0.747258	1.606605
	10.750		9.056582

← **Return to flat-rate valuation**

Appendix map

Appendix A25: OIS-curve equipment-financing audit

Year	Payment	Spot z_t	$e^{-z_t t}$	PV contribution
1	2.150	4.0381%	0.96042	2.06491
2	2.150	4.1924%	0.91957	1.97708
3	2.150	4.2201%	0.88108	1.89433
4	2.150	4.2397%	0.84401	1.81463
5	2.150	4.2741%	0.80759	1.73631
	10.750			9.48726

← **Return to curve valuation**

Appendix map

Appendix A26: Break-even values are numerical roots

Define the flat-rate residual

$$f(r) = 2.15 \sum_{t=1}^{5} (1+r)^{-t} - 9.20.$$

- $f(0) = +1.55$: instalments cost more with no discounting.
- $f(0.12) < 0$: instalments cost less at 12%.
- Monotonicity implies one crossing; bisection or Brent's method gives $r^* = 5.4250988\%$.
- Verify the root by checking that $|f(r^*)|$ is close to machine precision.

← **Return to break-even analysis**

Appendix map

Appendix A27: Optional rate sensitivity extension

For $V(y) = \sum_t P_t(1+y)^{-t}$,

$$\frac{dV}{dy} = -V D_{\text{mod}}, \qquad \frac{\Delta V}{V} \approx -D_{\text{mod}} \Delta y.$$

Macaulay duration is the PV-weighted payment date; modified duration converts it to first-order price sensitivity. A second-order correction uses convexity:

$$\frac{\Delta V}{V} \approx -D_{\text{mod}} \Delta y + \frac{1}{2}\mathcal{C}(\Delta y)^2.$$

These tools are developed with bond pricing on Day 2; here they provide an optional robustness extension.

Macaulay duration = PV-weighted payment date; modified duration = first-order percentage price sensitivity; convexity = second-order curvature adjustment.

← **Return to the decision memo**

Appendix map

Appendix A28: Notation, references, and data

- C_t: signed cash flow; $P_t = -C_t > 0$: payment-cost magnitude; d_t: discount factor.
- y_t: annual effective spot yield; z_t: continuously compounded spot yield; $f_{a,b}$: forward rate.
- Brealey, Myers, Allen and Edmans, *Principles of Corporate Finance*, Chs. 2–3.
- Berk and DeMarzo, *Corporate Finance*, Chs. 3–5.
- EconGraphs finance materials: consulted for visual sequencing of intertemporal exchange and present-value streams; all analytical figures in this deck are original.
- Bank of England: fitted sterling OIS curves and Bank Rate series IUDBEDR.
- Office for National Statistics: CPI 12-month inflation series D7BT.
- All calculations were independently reproduced from dated cash flows; the equipment-financing and retrofit examples are constructed cases.

Appendix map

Python practical map

1. P1 · Compound and discount one cash flow →
2. P2 · Calculate NPV with aligned arrays →
3. P3 · Build a loan amortisation schedule →
4. P4 · Convert APR, EAR, and period rates →
5. P5 · Value payments with the frozen OIS curve →
6. P6 · Solve and verify a break-even rate →

Use the supplied Day 1 notebook and frozen CSV data; amounts are in pounds unless a code comment states GBP million.

Technical appendix Lecture synthesis

Practical P1: Compound forward, then discount back

- Store the declared cash flow, rate, and horizon separately.
- Compute the year-3 value and recover the original date-0 value.

```
# Inputs use decimals: 4.5% becomes 0.045.
cash_today = 8000.0
annual_rate = 0.045
years = 3

future_value = cash_today * (1 + annual_rate) ** years
present_value = future_value / (1 + annual_rate) ** years
print(round(future_value, 2), round(present_value, 2))
```

`=` assigns a value; `**` means exponentiation; `round(x, 2)` displays two decimal places. Expected output: `9129.33 8000.0`.

← **Return to compounding**

Practical P2: NPV is a dot product of aligned arrays

- Create one array for dates and one for signed cash flows.
- Build discount factors in the same order, then multiply and add.

```
import numpy as np

times = np.array([0, 1, 2, 3])
cash_flows = np.array([-1000.0, 300.0, 450.0, 500.0])
discount_factors = 1 / (1 + 0.06) ** times
npv = np.dot(discount_factors, cash_flows)
print(round(npv, 2))
```

NumPy stores numerical arrays; `np.dot(a, b)` multiplies matching entries and sums them. Expected NPV: `103.33`.

← **Return to value additivity**

Practical map

Practical P3: Build and audit a loan schedule

- Calculate the contractual payment once at full precision.
- Update interest, principal repayment, and balance period by period.

```python
import pandas as pd

loan, rate, years = 20000.0, 0.06, 4
payment = loan * rate / (1 - (1 + rate) ** (-years))
balance, rows = loan, []

for year in range(1, years + 1):
    interest = rate * balance
    principal = payment - interest
    balance = balance - principal
    rows.append([year, interest, principal, balance])

schedule = pd.DataFrame(rows, columns=["year", "interest", "principal", "balance"])
```

pandas `DataFrame` makes a labelled table; a `for` loop repeats the indented update; `append` stores each row. The final balance should be numerically close to zero.

← **Return to loan payments**

Practical map

Practical P4: Convert rate quotations before valuation

- Convert a monthly-compounded APR into its monthly rate and EAR.
- Derive an equivalent quarterly rate from the same annual price.

```
apr = 0.12
periods_per_year = 12

monthly_rate = apr / periods_per_year
ear = (1 + monthly_rate) ** periods_per_year - 1
quarterly_rate = (1 + ear) ** (1 / 4) - 1

print(round(ear, 6), round(quarterly_rate, 6))
```

Variable names retain the quotation convention. Expected output: EAR `0.126825`; equivalent quarterly rate `0.030104`.

← **Return to rate conversion**

Practical P5: Read the OIS curve and align maturities

- Load the dated 12 August 2026 case-data CSV.
- Select annual maturities, convert percentages to decimals, and price the five payments.

```
import numpy as np
import pandas as pd

curve = pd.read_csv("day01_ois_curve_2026-08-12.csv")
annual = curve[curve["maturity_years"].isin([1, 2, 3, 4, 5])]
years = annual["maturity_years"].to_numpy()
spot = annual["spot_cc_pct"].to_numpy() / 100
discount_factors = np.exp(-spot * years)
pv = np.dot(discount_factors, np.full(5, 2.15))
print(round(pv, 4))  # GBP million
```

CSV = comma-separated values; `pd.read_csv` loads the file; `np.exp(x)` computes e^x; `isin` selects the declared maturities. Expected PV: `9.4873`.

← **Return to curve valuation**

Practical map

Practical P6: Solve and verify the financing threshold

- Define the valuation gap: instalment PV minus the cash price.
- Find the rate where that gap is zero and verify the residual.

```
from scipy.optimize import brentq

def gap(rate):
    instalment_pv = 2.15 * sum((1 + rate) ** (-t) for t in range(1, 6))
    return instalment_pv - 9.20

break_even = brentq(gap, 0.0, 0.12)
assert abs(gap(break_even)) < 1e-10
print(round(100 * break_even, 4))
```

A function packages a repeated calculation; `brentq` finds a root inside a sign-changing bracket; `assert` tests the result. Expected break-even rate: `5.4251` percent.

← **Return to the threshold**

Practical map