Properties of Stock Options

Lecture 10

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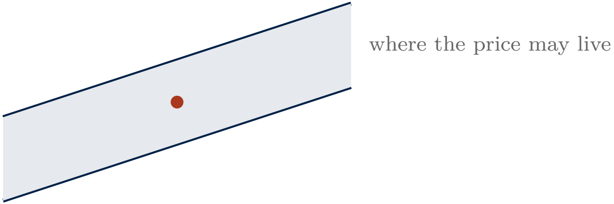

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Outline

1. No-arbitrage bounds
2. Put-call parity
3. Early exercise and dividends

No-arbitrage bounds

One tool, applied a ninth time

Topic 9 left two questions: a law hiding on the board, and whether early exercise is ever worth anything. Both are settled tonight with the course's oldest tool:

two portfolios with identical payoffs must have identical prices.

No-arbitrage priced gold (Lecture 1), the curve through time (Lecture 4), storage and carry (Lecture 5), and the swap by decomposition (Lecture 7). Tonight it bounds **option** prices — with no model of how the underlying moves, and no statistics. *Main point: before any pricing model, no-arbitrage alone fixes where an option price may and may not sit.*

Notation and assumptions

- c, p for European, C, P for American; spot $S = 4.000$, rate $r = 2\%$, maturity $T = 2/12$; the June board throughout.
- No dividends until Block C. (The local ETF options adjust for distributions anyway, so the clean case is also the realistic one here.)
- Proofs are frictionless, but a real board obeys the laws only *to within the bid–ask spread* (0.002 here) — keep that tolerance for every audit.

Six factors that move a premium

factor (intuition; the laws follow)	call	put

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strike K higher	$\downarrow$	$\uparrow$
time T longer (American)	$\uparrow$	$\uparrow$
volatility σ rises	$\uparrow$	$\uparrow$
rate r rises	$\uparrow$	$\downarrow$
dividends D larger	$\downarrow$	$\uparrow$

The European-T row is not guaranteed (Block C); the r and D rows are Lecture 5's carry.

Why volatility raises both calls and puts

- The kink keeps the favourable tail and discards the unfavourable one: more spread in outcomes adds value in the tail the holder keeps, while the tail discarded was never the holder's loss.
- A symmetric instrument is indifferent to spread (a forward's expected payoff does not depend on it); a **kinked** instrument feeds on it — the same reason a tranche depended on the loss distribution's shape (Lecture 8).

Main point: an option is long volatility, and pricing exactly what that is worth is the task of the rest of Part II.

The two-portfolio argument: a ceiling

Each law is proved the same way: build two portfolios, carry them to expiry, and compare in *every* state. First, the call's ceiling $c \leq S$:

at expiry	$S_T \leq K$	$S_T > K$
A: one ETF unit	S_T	S_T
B: the call	0	$S_T - K$
verdict	A $\geq$ B	A $>$ B

A never loses, so A cannot cost less today: $S \geq c$. The right to buy a thing cannot be worth more than the thing.

The call's floor

Portfolio A: the call plus Ke^{-rT} lent at r. Portfolio B: one ETF unit.

at expiry	$S_T \le K$	$S_T > K$
A: call + loan matured to K	K	S_T
B: the ETF unit	S_T	S_T
verdict	A $\geq$ B	A $=$ B

So $c + Ke^{-rT} \geq S$, i.e. $c \geq S - Ke^{-rT}$. Audit at $K = 3.8$: the floor is $4 - 3.7874 = 0.2126$; the board shows 0.240, lawful, riding **0.0274** above its floor. Hold that gap.

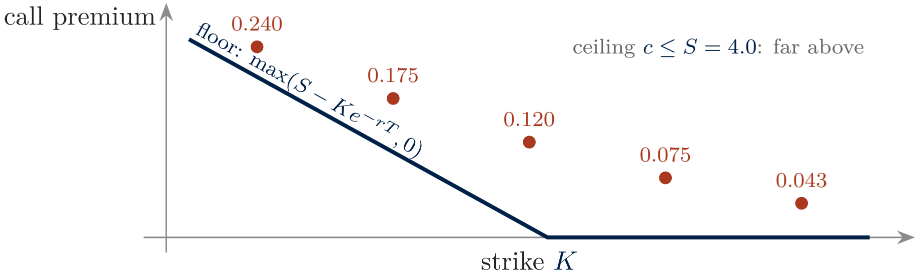
The put's floor, mirrored

Portfolio A: the put plus one ETF unit. Portfolio B: Ke^{-rT} lent at r.

at expiry	$S_T \le K$	$S_T > K$
A: put + the ETF unit	K	S_T
B: loan matured to K	K	K
verdict	A = B	A ≥ B

So $p + S \ge Ke^{-rT}$, i.e. $p \ge Ke^{-rT} - S$. Audit at $K = 4.2$: the floor is $4.1860 - 4 = 0.1860$; the board shows 0.229, lawful, riding **0.043** above. Hold that gap too.

The corridor, drawn



Five premiums, all floating lawfully above the floor by a margin that shrinks, peaks, then shrinks. Note the shape of those margins.

The American corrections

Freedom is weakly positive (Topic 9): $C \geq c$ and $P \geq p$ always. The ceilings shift:

- $C \leq S$ still — the right to buy never beats owning.
- European put: $p \leq Ke^{-rT}$ (its best case, K, arrives only at T, discounted).
  American put: $P \leq K$ — the best case can be taken today.

The gap between Ke^{-rT} and K — worth $K(1 - e^{-rT}) = 0.0133$ here at $K = 4.0$ — is where Block C's drama lives.

Auditing the board

Run every Block A law across all ten cells of the June board:

call ceilings ($c \leq 4.000$)	five cells, five passes
put ceilings ($p \leq Ke^{-rT}$)	five cells, five passes
call floors ($c \geq S - Ke^{-rT}$)	five cells, five passes
put floors ($p \geq Ke^{-rT} - S$)	five cells, five passes

Twenty checks, twenty passes: the board is law-abiding. But these fences are *loose* — they say where prices may not go, not where they must sit. The board obeys something tighter.

Why a violation is never seen

A premium outside the corridor is a no-capital free lunch: buy the cheap side of the comparison, sell the dear side, collect today, owe nothing in any state.

- Market makers run these checks by machine, continuously, so violations die in milliseconds.
- Lecture 1's point holds: arbitrage is self-erasing, so the absence of free lunches is the evidence of enforcement, not of impossibility.

And when your own screen seems to show a violation, the desk note (frame 39) has a rule.

The margins that file in columns

The margins above the call floor (frame 10), set beside the put column for the in-the-money strikes (where the floor is alive):

strike	call's height above floor	the put's price

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4.0	0.1067	0.106

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To within a fen, the call's height above its floor *is* the put. The mirror holds too: the puts' heights above their floors, 0.0746 and 0.0430, are the 4.1 and 4.2 calls. This is not coincidence.

Where we are

the corridor, after Block A

the two-portfolio argument, all states, one verdict	✓
ceilings $c \le S$, $p \le Ke^{-rT}$ (American $P \le K$)	✓
floors $c \ge S - Ke^{-rT}$, $p \ge Ke^{-rT} - S$	✓
the board: twenty checks, twenty passes	✓
the margins that match the other wing	unexplained
early exercise; dividends	Block C

Block B: the two floor proofs were secretly the same, and the board has been keeping a law it was never taught.

Put-call parity

Two portfolios, one payoff

Look again at the winning portfolios of frames 8 and 9:

- the call plus Ke^{-rT} lent $\longrightarrow$ worth $\max(S_T, K)$ at expiry;
- the put plus the ETF unit $\longrightarrow$ worth $\max(S_T, K)$ at expiry.

Not similar, not dominant: **identical in every state** — two recipes for the same payoff. And the course has one rule for that, nine lectures old.

Two recipes for $\max(S_T, K)$

- **Recipe 1 — call plus reserve**: hold the call and park Ke^{-rT} in bonds. If the ETF rises, the call collects it; if it falls, the reserve matures to K.
- **Recipe 2 — protected portfolio**: hold the ETF and a put. If the ETF rises, you own the gain; if it falls, the put sells it at K.

Recipe 2 is exactly the protective-put floor of frame 3. Recipe 1 is its twin: cash plus calls.

Put-call parity

Same payoff in every state implies the same price today:

$$\boxed{c + Ke^{-rT} = p + S}$$

at expiry	$S_T \leq K$	$S_T > K$
call + loan matured	K	S_T
put + ETF unit	K	S_T
verdict	equal	equal

Main point: this is an identity, not an inequality — four instruments, one equation, holding for any distribution of S_T and with no model anywhere.

The board obeys it

Topic 9's board, tested ($c - p$ against $S - Ke^{-rT}$):

strike	$c - p$	$S - Ke^{-rT}$	gap

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3.8	$+0.2130$	$+0.2126$	$+0.0004$

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3.8	+0.2130	+0.2126	+0.0004
3.9	+0.1130	+0.1130	0.0000
4.0	+0.0140	+0.0133	+0.0007

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4.1	-0.0860	-0.0864	$+0.0004$

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4.1	-0.0860	-0.0864	$+0.0004$
4.2	-0.1860	-0.1860	0.0000

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4.1	-0.0860	-0.0864	$+0.0004$
4.2	-0.1860	-0.1860	0.0000

Main point: every gap is inside the bid–ask spread — the board was obeying a law before it was stated; finance audits reality, it does not legislate it.

The matching margins, explained

Rearrange the identity:

$$c - (S - Ke^{-rT}) = p \qquad \text{and} \qquad p - (Ke^{-rT} - S) = c.$$

- Each wing's height above its own floor is exactly the other wing's price — frame 14's columns were parity, displaying itself.
- Topic 9's moneyness diagonal was geography; this is anatomy — the two halves of the board are one object, written twice.

Main point: inside every call is a put — a call equals a put plus a financed position in the asset — which is why the early-exercise argument later is literal, not a metaphor.

Synthetics: any three make the fourth

synthetic call	put + ETF − borrowing of Ke^{-rT}
synthetic put	call − ETF + lending of Ke^{-rT}
synthetic ETF	call − put + lending of Ke^{-rT}
synthetic bond	ETF + put − call (the *conversion*)

So the **protective put** equals **bills plus calls** — identical payoff by law — and Topic 11's choice between them turns on frictions and mandate, never on payoff.

The conversion arbitrage

Suppose a dealer misquotes the 4.0 call at **0.140** (its parity value is 0.1193). Sell it and buy the replicating recipe:

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sell the call	$+0.140$
buy the put	-0.106
buy the ETF unit	-4.000
borrow Ke^{-rT}	$+3.987$

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Suppose a dealer misquotes the 4.0 call at **0.140** (its parity value is 0.1193). Sell it and buy the replicating recipe:

sell the call	$+0.140$
buy the put	-0.106
buy the ETF unit	-4.000
borrow Ke^{-rT}	$+3.987$
collect today	$+0.0207$
owe at expiry, in every state	0

A riskless ¥207 per contract. Machines running exactly this are why the frame-19 gaps were fractions of a fen.

The frictions corridor

Why "to within the spread"? Because the recipe pays only after costs:

- spreads on three legs, fees, and — decisively — the **shorting constraint** (Lecture 5): half the recipes require selling an ETF one may not hold.
- So parity in practice is a corridor whose width is the price of those frictions. A persistent one-sided gap is not free money; it is the constraint, quoted — Lecture 5's index-futures discount, on the options board.

Main point: where one leg of the arbitrage is constrained, a persistent "parity violation" is the priced cost of the constraint, not an opportunity.

Case: the 2008 short-sale ban

In September 2008, after Lehman and AIG, US regulators banned short sales in **about 800 financial stocks** overnight (19 September; lifted 8 October).

- Bearish demand moved to the only route left — *synthetic* shorts (buy puts, sell calls) — and apparent put-call parity violations in the banned names widened sharply.
- These were largely not free lunches: with the stock impossible to short, the market maker could not hedge, and the "violation" was mostly the price of that constraint (Battalio and Schultz, 2011).

When a law appears broken, first price the constraint.

Parity also quotes the forward

Rewrite parity with the forward $F_0 = Se^{rT}$ (Lecture 5's carry):

$$c - p \;=\; (F_0 - K)\, e^{-rT}.$$

- Call and put are equal exactly at $K = F_0$, so the call–put gap changes sign at the forward.
- On the board, $c > p$ at 4.0 and $c < p$ at 4.1; the crossing sits near $4.013 = 4.000\, e^{0.02\times 2/12}$.

Main point: the option board quietly quotes the forward price — Part I and Part II are one machine, the kink drawn on top of the carry.

American parity is a band

With early exercise possible, the identity loosens to a band:

$$S - K \ \leq \ C - P \ \leq \ S - Ke^{-rT}.$$

- Its width is $K(1 - e^{-rT})$ — the interest on the strike, 0.0133 here at $K = 4.0$.
- The looseness has one cause: someone might exercise early. Pin down when that is rational and the band tightens — which is Block C.

The protective put, certified

The audit from frame 3, complete:

- bounds: twenty checks, twenty passes (frame 12);
- parity: five strikes inside the spread (frame 19);
- forward consistency: the call–put crossing where Lecture 5 puts it (frame 25).

The protective-put quote is **lawfully priced**: there is no cheaper synthetic route and no mispricing to harvest. Whether to *buy* the floor is a strategy decision (Topic 11), but the price is fair relative to every other instrument on the board.

Early exercise and dividends

From a rule to a theorem

Topic 9's practical rule — sell an option rather than exercise it — is now promoted:

> For an American **call** on an asset paying no income, early exercise is never optimal: $C = c$.

No model and no probabilities; the proof is one line of Block A, with three reasons to remember.

The one-line proof

At any time before expiry, with $\tau > 0$ remaining:

$$C \;\geq\; c \;\geq\; S - Ke^{-r\tau} \;>\; S - K \;=\; \text{the value of exercising now.}$$

- The live call is always worth strictly more than the value of exercising — the floor of frame 8 already said so.
- To exit, **sell** the call (collecting at least the floor); never exercise it (collecting only $S - K$).

Main point: an American call on a non-income asset is worth more alive than exercised, so it is never exercised early — proved with no new machinery.

Three reasons

Exercising a call early:

- **pays K too soon**, surrendering the interest on the strike — the 0.0133 the American band was guarding;
- **discards the insurance**: parity (frame 20) showed a put living inside every call, so exercising takes on the whole downside the live call never carried;
- **collects nothing**: a non-income asset pays nothing while you wait, so there is nothing on the other side of the scale.

Interest kept, insurance kept, nothing missed — waiting wins on all three.

The deepest call

Test the theorem where instinct says "take the money": an American call struck at $K = 3.0$, ETF at 4.0, two months left.

exercise now	$4.000 - 3.000 = 1.000$

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exercise now	$4.000 - 3.000 = 1.000$
live floor: $4.000 - 3.0\,e^{-rT}$	1.010
selling beats exercising by at least	¥100 per contract

Even at its deepest, the market must bid above the exercise value — frame 8’s argument guarantees a buyer at the floor. Topic 9’s ¥50 offset gap was this theorem at work.

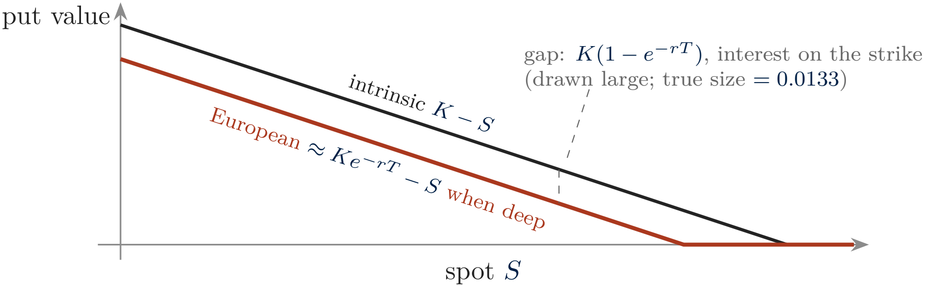
Puts break the symmetry

Run the three reasons for an American **put**:

- exercising early **receives** K sooner, so interest now works *for* acting;
- the insurance argument fades — once the ETF has fallen far, little upside remains to protect;
- still nothing is collected by waiting.

At the extreme, with the asset near zero, waiting is pure cost: the most a put delivers is K, available today. So **American puts can be optimal to exercise early when deep enough in the money**, and their freedom is genuinely worth paying for: $P > p$.

Negative time value, explained



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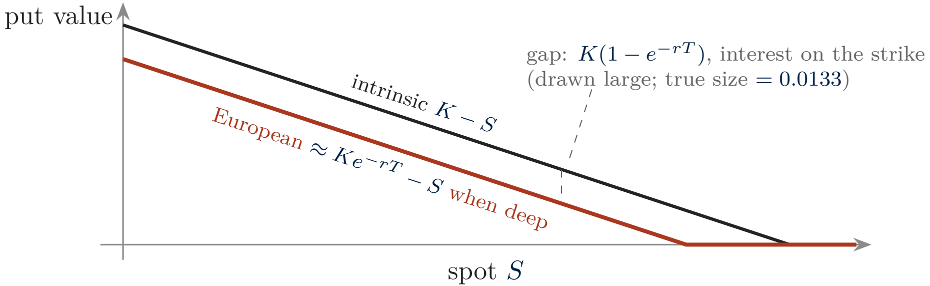

Main point: a deep European put trades below intrinsic — its time value goes to $-K(1 - e^{-rT})$ *— because the holder is forbidden to take* K *early; the American put never sinks below* $K - S$.

Pricing the put's freedom

Suppose a dealer offers an *American*-style 3.8 put — "more flexible" — at a premium above the listed 0.027.

- In principle this is fair: put freedom is genuinely valuable ($P > p$).
- In this case the put is out of the money with two months to run, so rational early exercise is far away and the privilege should be worth near zero.
- How near, exactly? The bounds cannot say — pricing the *interior* of the corridor needs a model of movement. Topic 12's trees price the American put; until then, a large "flexibility" charge on an out-of-the-money short-dated put is declined.

Dividends

In markets that do not adjust contracts (classic US stock options), a dividend of present value D leaves the price before expiry. Every law patches the same way — replace S by $S - D$:

$$c \geq S - D - Ke^{-rT}, \qquad p \geq D + Ke^{-rT} - S, \qquad c - p = S - D - Ke^{-rT}.$$

Worked: $D =$ present value of a 0.04 payout next month $= 0.0399$. The at-the-money call's parity value falls by four fen and the put gains the same.

The dividend exception

Dividends break the third reason of frame 30: now waiting *does* miss something. For a deep-in-the-money American call, exercising just before the ex-dividend moment captures D, which pays when

$$D > K\left(1 - e^{-r\tau}\right) \qquad (\tau = \text{time after the ex-date}).$$

Worked: $D = 0.04$ against $4.0 \times (1 - e^{-0.02/12}) = 0.0067$ — the dividend out-earns the surrendered interest several times over, so early exercise can be rational, at the dividend and never between.

- Where contracts adjust for the dividend (the local ETF options), the adjustment restores the theorem: one convention, one theorem; another convention, one exception.

What the laws give, and what they cannot

the corridor, complete

ceilings and floors, by the two-portfolio argument	✓
parity — the identity, caught live on the board	✓
synthetics, the conversion, the frictions corridor	✓
the theorem: calls are not exercised early; puts may be; dividends patched	✓
the exact price *inside* the corridor	needs a model of movement

Main point: no-arbitrage fences the price into a corridor; a model is needed to locate it inside — bounds tonight, models for the rest of Part II.

Summary

- Lectures 1–5 priced symmetric promises by no-arbitrage; Lecture 7 decomposed the swap by it; Lecture 9 introduced the option.
- Lecture 10: **even choice obeys the law of one price.**

Option prices felt exotic, but they price like everything else in the course — by building the payoff two ways and refusing to pay two prices. The kink changed the payoff, never the law: no-arbitrage gives a corridor and one exact identity (put-call parity), and a model is needed only to locate the price inside.

A desk note

the audit	the board is certified — bounds, parity, and the forward crossing all within the spread; the protective-put quote is lawfully priced
synthetics	protective put $\equiv$ bills plus calls by law; choose on frictions, never on payoff
exercise policy	calls on non-income assets: sell, never exercise — now a theorem; ex-dividend dates matter only in unadjusted markets
flexibility quotes	American-put premiums challenged on moneyness and tenor; large charges on out-of-the-money short-dated puts declined
apparent violations	data-check first, phone call second — almost every "arbitrage" a junior finds is a stale quote, a dividend, or a borrow cost

In-class problems

A. The June 3.8 call is quoted at 0.190. Which law does it break? Write the four-leg trade and its riskless profit per contract, and name the friction that could spoil it.

B. With the 4.0 call at 0.120, a dealer quotes the 4.0 put at 0.095. Compute the lawful put price, the trade, and the profit per contract.

C. An American call, $K = 3.5$, ETF at 4.2, three months, no income. A colleague urges "lock in the 0.70 now." Refute with one inequality and one sentence.

What carries into Part II

- **synthetics** are the building blocks of Topic 11's spreads, collars, and the protective-put decision;
- the corridor's **interior**, and "how deep is deep enough" for an American put, need Topic 12's trees;
- the **volatility** factor — the one no one can observe directly — is priced by Topic 13's formula, which fills the IV column;
- **put-call parity** becomes the first sanity check on every price the course produces from here on.

The prices obey the laws, and the laws leave room.

Next: building inside the lawful set — spreads, collars, covered calls — and choosing the floor.

Topic 11: trading strategies involving options.