Interest Rates

Lecture 4

https://fatih.ai/sjtu/derivatives/

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Outline

1. What a quoted rate really says
2. The zero curve: a price for every date
3. When the curve moves

One ticket, two choices

Loto-Québec, *Gagnant à vie*, 2025.

In 2025, **Brenda Aubin-Vega**, aged 20, from Montreal, scratched a Loto-Québec ticket and won the top prize. She had to choose:

option A	\$1,000,000, paid today
option B	\$1,000 every week, for life

She said no to the million. Many people online called her a fool.
Vote: was she right?

To answer, you need three things. This lecture builds all three, and by the end of today you will price this prize.

1. what a quoted interest rate really means (compounding)
2. a price of money for *every* future date (the zero curve)
3. what happens to a value when rates move (duration)

What a quoted rate really says

Same problem: compare bank offers

Two banks offer you a one-year deposit for ¥1,000,000:

Bank A	3.00%, paid once a year
Bank B	2.98%, paid quarterly

Bank B's quarterly interest itself earns interest:

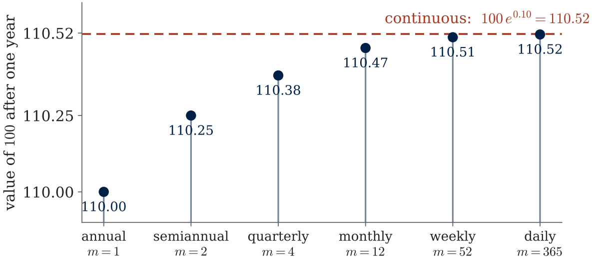
$$\left(1 + \frac{0.0298}{4}\right)^4 - 1 \;=\; 3.013\% \;>\; 3.000\%.$$

The smaller printed rate pays more.

— *A rate means nothing without its compounding convention. Convert to a common basis before you compare.* —

The compounding limit

¥100 at "10%" for one year, compounded m times, grows to $100\left(1+\frac{0.10}{m}\right)^m$:



KEY EQUATION · CONTINUOUS COMPOUNDING

$$A\left(1+\frac{R}{m}\right)^{mT} \xrightarrow{m\to\infty} A\,e^{RT}$$

Daily compounding is already almost continuous. From here the course quotes every rate continuously: the upgrade promised in Lecture 1.

Converting between conventions

ROUTE 1: m TIMES A YEAR $\rightarrow$ CONTINUOUS

$$A\,e^{R_c n} = A\left(1 + \frac{R_m}{m}\right)^{mn} \qquad \text{same growth}$$

$$e^{R_c n} = \left(1 + \frac{R_m}{m}\right)^{mn} \qquad \text{divide by } A$$

$$e^{R_c} = \left(1 + \frac{R_m}{m}\right)^{m} \qquad \text{per year, } n = 1$$

$$R_c = m\,\ln\left(1 + \frac{R_m}{m}\right) \qquad \text{ln both sides}$$

ROUTE 2: CONTINUOUS $\rightarrow$ m TIMES A YEAR

$$\frac{R_c}{m} = \ln\left(1 + \frac{R_m}{m}\right) \qquad \text{divide by } m$$

$$e^{R_c/m} = 1 + \frac{R_m}{m} \qquad \ln x = y \Rightarrow x = e^y$$

$$\frac{R_m}{m} = e^{R_c/m} - 1 \qquad \text{isolate } R_m/m$$

$$R_m = m\left(e^{R_c/m} - 1\right) \qquad \text{multiply by } m$$

KEY EQUATION · CONVERTING CONVENTIONS

$$R_c = m\,\ln\left(1 + \frac{R_m}{m}\right) \qquad\qquad R_m = m\left(e^{R_c/m} - 1\right)$$

Converting between conventions

A bank quotes 10% with semiannual compounding ($m = 2$). In continuous terms,

$$R_c = 2\ln(1 + \tfrac{0.10}{2}) = 2\ln(1.05) = 2 \times 0.04879 = \mathbf{9.758}\%.$$

Same growth, smaller printed number: only the unit changed, like kilometres and miles.

Check yourself — Convert 8% with monthly compounding to continuous.

Answer: — $12\ln(1 + 0.08/12) = 7.97\%$

Which rate? Many borrowers, many prices

Even on one convention there are many rates, because there are many borrowers of different credit quality:

	what it is	example
government bonds	the sovereign borrowing in its own currency	CGBs here, Treasuries abroad
repo	cash lent against collateral, close to risk-free (Lecture 2)	DR007, SOFR
bank benchmarks	the reference rates floating loans reset on	LPR, SHIBOR

The **credit spread** is the extra rate a riskier borrower must pay for default risk. Derivatives desks discount at rates built from overnight benchmark markets, not at Treasury yields.

Benchmark reform, two countries

	United States / UK	China
submission-based	**LIBOR**: a panel's opinion of its own borrowing cost	**SHIBOR**: panel quotes, unsecured
transaction-based	**SOFR**: measured from actual Treasury repo (NY Fed, 2018)	**DR007**: actual 7-day repo among banks
what loans reset on	term SOFR and spreads	**LPR**: 3.0% / 3.5% (June 2026)

- LIBOR failed twice: banks manipulated their submissions for profit (fines above $9 billion), and after 2008 the market it described mostly stopped trading. The last synthetic dollar settings ended 30 September 2024.
- Timing changed too: LIBOR was known at the *start* of each period; SOFR-type rates are compounded from realised overnight rates, known only at the *end*.

— *The reform direction is the same everywhere: from opinions about a market to measurements of one.* —

The zero curve: a price for every date

Money has a price for every date

Definition — The **zero rate** $z(T)$ is the continuously compounded rate on money lent now and repaid in a single amount at T, with nothing in between.

Earlier R was one continuously compounded rate. Now the rate depends on the date: $z(T)$ is a rate, and T is time in years.

KEY EQUATION · DISCOUNT FACTORS

$$\text{¥1 today} \longrightarrow e^{z(T)\,T} \text{ at } T \qquad \text{¥1 at } T \longrightarrow e^{-z(T)\,T} \text{ today}$$

$$PV_0 = CF_T\, e^{-z(T)T}$$

CF_T is the cash flow paid at date T. The exponent is “rate × time”: $z(T)T$. The minus sign discounts future money back to today.

- If $z(1) = 2.3\% = 0.023$, then ¥1 in one year is worth $e^{-0.023} = ¥0.977$ today.
- The **zero curve** is the full list: $z(0.5), z(1), z(1.5), \ldots$

Where do the zero rates come from?

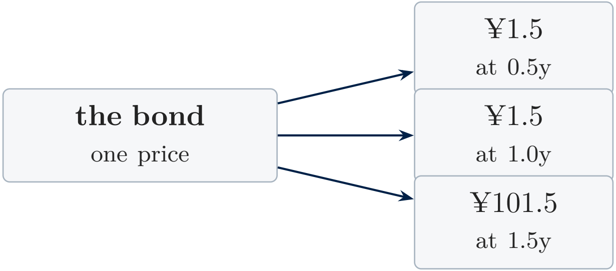
The market gives us some dates directly, but not the whole curve.

deposit	one payoff at maturity, so the rate is direct: a 6-month deposit gives $z(0.5)$; a 1-year deposit gives $z(1)$
coupon bond	many payoffs on different dates, so the price mixes several discount factors

— *Bootstrapping means using observed bond prices to solve for the next missing zero rate, one maturity at a time.* —

A bond is a bundle of dated cash flows

A 1.5-year government bond, 3% coupon paid semiannually, is three separate promises:



Its price is the sum of the three, each multiplied by its discount factor:

$$P = 1.5\,e^{-z(0.5)\cdot 0.5} + 1.5\,e^{-z(1)\cdot 1} + 101.5\,e^{-z(1.5)\cdot 1.5}.$$

Every value in this course is cash flows times discount factors, summed. The curve supplies the discount factors.

Bootstrap idea: build from near to far

Bootstrapping is a recursive way to build the zero curve.

start	short deposits give the first points: $z(0.5) = 2.0\%$ and $z(1) = 2.3\%$
use a bond	choose a bond whose last payment is the next date you want, here 1.5 years
strip	price the earlier payments with known rates, subtract them from the bond price, and solve for the missing rate

— *The focus is not the bond itself. The bond is evidence used to infer the next point on the zero curve.* —

Strip out the known cash flows

The 1.5-year bond trades at **¥100.715**. Its first two payments can already be priced using $z(0.5)$ and $z(1)$:

cash flow	discount factor	value today
¥1.5 at 0.5y	$e^{-0.02\times0.5}$	¥1.485
¥1.5 at 1.0y	$e^{-0.023\times1}$	¥1.466
known value		¥2.951

$$\underbrace{100.715}_{\text{bond price}} - \underbrace{2.951}_{\text{known pieces}} = \underbrace{97.764}_{\text{value today of final payment}}.$$

The remaining ¥97.764 must be today's value of the final ¥101.5 paid at 1.5 years.

Extract the missing rate

The final payment is ¥101.5 at 1.5 years. Its value today is ¥97.764:

$$
\begin{aligned}
101.5\, e^{-1.5\, z(1.5)} &= 97.764 && \text{future cash flow} \times \text{discount factor} \\
e^{-1.5\, z(1.5)} &= \frac{97.764}{101.5} = 0.96319 && \text{the 1.5-year discount factor} \\
-\,1.5\, z(1.5) &= \ln(0.96319) && \text{undo the exponential} \\
z(1.5) &= -\frac{\ln(0.96319)}{1.5} = \mathbf{2.50\%} && \text{the new zero rate}
\end{aligned}
$$

This is bootstrapping: known earlier rates plus one bond price produce the next zero rate.

Your turn: the two-year point

> YOUR TURN — A 2-year bond pays a 3.4% coupon semiannually (¥1.7 each half-year) and trades at ¥101.336. Find $z(2)$. You know $z(0.5), z(1), z(1.5)$.

Strip the three known coupons:

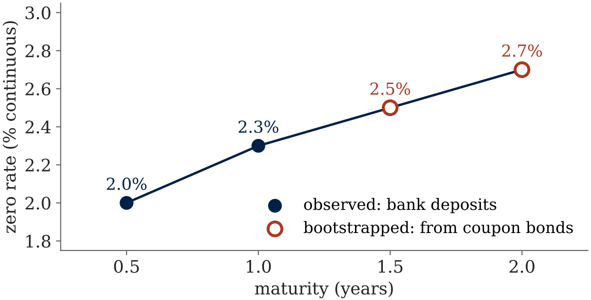
$$101.336 - 1.683 - 1.661 - 1.637 = 96.354, \qquad \text{so} \qquad 101.7\, e^{-2\, z(2)} = 96.354.$$

Divide, then take ln:

$$e^{-2\, z(2)} = 0.94743 \quad \Longrightarrow \quad z(2) = -\frac{\ln(0.94743)}{2} = \mathbf{2.70}\%.$$

Note the recursion: this step used every point below it, including the one you built a minute ago.

The curve, built



T	$z(T)$	source
0.5	2.0%	deposit quote
1.0	2.3%	deposit quote
1.5	2.5%	1.5-year bond price
2.0	2.7%	2-year bond price

- This is not a history of rates. It is today's price list for money at future dates.
- For any cash flow at date T, use the curve to get the discount factor $e^{-z(T)T}$.
- For dates between quoted points, desks interpolate; for later dates, they keep bootstrapping with longer instruments.

Yield and par yield: two summaries, not the curve

A **yield** compresses one bond into one number. It asks: what single flat rate y would reproduce this bond's price if every cash flow used the same rate?

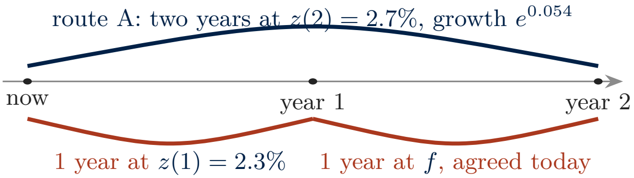
$$101.336 = 1.7\,e^{-0.5y} + 1.7\,e^{-y} + 1.7\,e^{-1.5y} + 101.7\,e^{-2y}.$$

- No algebra can isolate y here: there is no ln step. A solver iterates to $y = \mathbf{2.69}\%$.
- Change the coupon and the yield changes; the zero rates do not. The zero curve prices dates; a yield summarizes one bond.

A **par yield** answers a different question: what coupon rate would make a new bond sell for exactly ¥100? On our curve, the 2-year par yield is **2.71%** with semiannual coupons.

Yield = summary of an existing bond. Par yield = coupon on a new par bond. The par yield returns in Lecture 7 as the swap rate.

Forward rates: two routes through time



Same starting money, same year-2 destination. No-arbitrage (Lecture 1) forces the two growth routes to match:

$$e^{0.023}\, e^{f} = e^{0.054} \qquad \text{growth factors multiply}$$

$$f = 0.054 - 0.023 = \mathbf{3.1}\% \qquad \text{take ln: exponents add}$$

KEY EQUATION · FORWARD RATE BETWEEN T_1 AND T_2

$$f(T_1, T_2) = \frac{z(T_2)\, T_2 - z(T_1)\, T_1}{T_2 - T_1}$$

Where the forward formula comes from

We want the rate for the future interval T_1 to T_2, agreed today. The two routes must give the same money at T_2.

$$e^{z(T_2)T_2} = e^{z(T_1)T_1}\, e^{f(T_1,T_2)(T_2-T_1)} \qquad \text{direct route = split route}$$

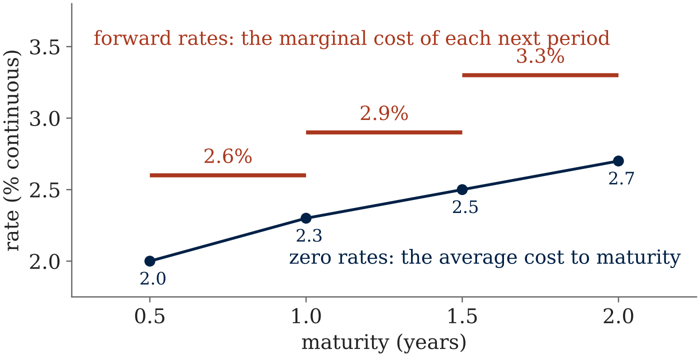
$$z(T_2)T_2 = z(T_1)T_1 + f(T_1,T_2)(T_2 - T_1) \qquad \text{take ln: exponents add}$$

$$z(T_2)T_2 - z(T_1)T_1 = f(T_1,T_2)(T_2 - T_1) \qquad \text{move the known first leg}$$

$$f(T_1,T_2) = \frac{z(T_2)T_2 - z(T_1)T_1}{T_2 - T_1} \qquad \text{divide by the future interval}$$

— *A zero rate is an average from today to a date. A forward rate is the rate for one future slice of time implied by those averages.* —

The forward staircase



- The zero rate is the *average* cost of money to T; the forward is the *marginal* cost of the next period.
- Zeros rise 2.0 to 2.7; forwards rise 2.6 to 3.3. A rising average needs a marginal rate above it, the same logic as marginal and average cost in microeconomics.

What is the benchmark in an FRA?

A floating-rate loan does not know its final interest rate today. The loan contract names a market reference rate, called the **benchmark**.

$$\text{loan rate in year 2} = \underbrace{\text{benchmark set in year 1}}_{\text{unknown today}} + 0.85\%.$$

- The benchmark is the floating market rate in the contract (SOFR, SHIBOR, DR007, LPR, or another named rate).
- Today's curve gives the rate Jade Steel can lock for year $1 \to 2$: $f = 3.1\%$ continuous, or $e^{0.031} - 1 = \mathbf{3.15}\%$ annual.
- The 0.85% is the bank's credit spread. The FRA hedges the benchmark part; it does not remove the spread.

The rows on the next slide are possible future benchmark outcomes: below, equal to, or above the locked 3.15%.

Locking a future rate: the FRA

A **forward rate agreement** fixes the future benchmark part of a floating loan. Jade Steel will borrow ¥100M for one year at benchmark $+0.85\%$. It uses an FRA to lock the benchmark at **3.15%**.

$$\text{target all-in rate} = 3.15\% + 0.85\% = \mathbf{4.00}\%.$$

future benchmark	loan interest	FRA cash flow	**all-in cost**
2.65%	¥3.50M	−¥0.50M	**¥4.00M**
3.15%	¥4.00M	¥0	**¥4.00M**
3.65%	¥4.50M	+¥0.50M	**¥4.00M**

Borrower's view: + means receive from the FRA, − means pay. Day counts ignored until Lecture 6.

— *An FRA struck at today's forward costs nothing to enter. Later, its cash flow offsets the benchmark surprise.* —

Marking an old FRA to market

Last year your firm agreed to *receive* a fixed 3.65% (annual) and pay the benchmark, on ¥100M, for year $1 \to 2$. Rates have fallen: today's forward is 3.15%. What is the old agreement worth now? Mark to market means: compare the old contract with the new zero-value contract available today, then discount the difference.

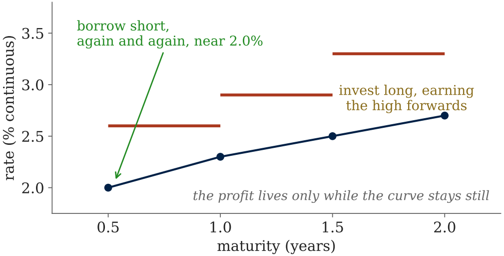
$$
\begin{aligned}
\text{a new FRA at } 3.15\% \text{ is worth } 0 & & \text{struck at today's forward} \\
\text{extra cash at year 2} &= (3.65\% - 3.15\%) \times ¥100\text{M} = ¥500{,}000 & \text{only the fixed rate differs} \\
\text{value today} &= 500{,}000 \times e^{-0.027 \times 2} & \text{discount 2 years at } z(2) \\
&= 500{,}000 \times 0.94743 = \mathbf{¥473{,}716} &
\end{aligned}
$$

Replace and discount: value an old contract by pricing the new one that would cancel it.

A forward is a breakeven, not a forecast

The 3.1% forward is the rate you can *lock today at zero cost.* It is not the market's prediction of where rates will be.

> ORANGE COUNTY, 1994 — Treasurer Robert Citron borrowed at short maturities and invested at long ones, collecting the higher forward rates. For two years the strategy made money. Then the Fed raised rates sharply; in December 1994 the county reported a \$1.5 billion loss and filed for bankruptcy.

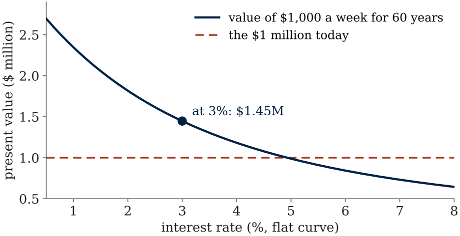


— A forward rate is a price you can trade, not a prediction you can trust. —

Price the prize

Our curve stops at two years, and its forwards climb: 2.6, 2.9, 3.3. For a 60-year stream you need the long end. Assume the curve flattens at **3%**, and that Brenda collects for 60 years (she is 20; Canadian women live to their mid-80s, so 60 years is slightly conservative). This is the same rule as before: each future payment gets turned into today's money. First convert the flow: \$1,000 a week ≈ \$52,000 a year. Then discount every year back at 3%:

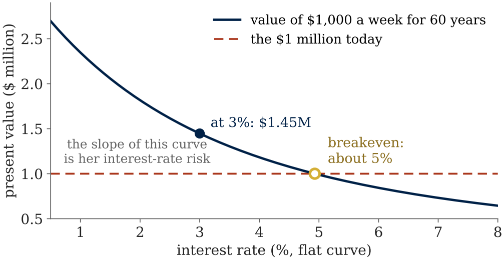
$$
\begin{aligned}
PV &= 52{,}000\,\frac{1-e^{-0.03\times 60}}{0.03} \qquad \text{add up } e^{-rt}\text{, 60 years} \\
&= 52{,}000 \times 27.82 \\
&= \$\mathbf{1.45}\text{M} > \$1\text{M}
\end{aligned}
$$



Undiscounted she collects about \$3.1M; discounting at 3% removes more than half. At today's rates, **she chose well**.

When the curve moves

The prize is a rate bet



- The value is not a number; it is a curve of answers. At about 5%, the weekly stream and the million are worth the same.
- Below 5% her choice wins; above it, the million wins: a bet that rates stay low, Orange County's mirror image.

> THE QUESTION OF THIS BLOCK — How do you *measure* the sensitivity of a value to the curve?

Duration: the slope of a bundle

For a bond priced off one flat yield y:

$$B = \sum_i c_i \, e^{-y t_i} \qquad \text{cash flows times discount factors}$$

$$\frac{dB}{dy} = -\sum_i t_i \, c_i \, e^{-y t_i} \qquad \text{differentiate: each term brings down } -t_i$$

$$= -B \sum_i t_i \, w_i, \qquad w_i = \frac{c_i e^{-y t_i}}{B} \qquad \text{factor out } -B\text{; the } w_i \text{ sum to 1}$$

KEY EQUATION · DURATION

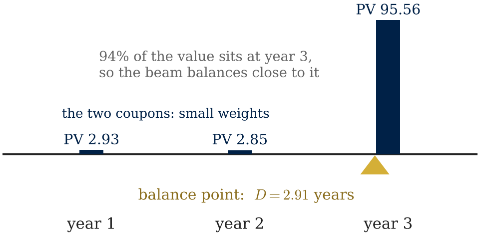
$$\Delta B \approx -B \, D \, \Delta y, \qquad D = \sum_i t_i \, w_i$$

D reads two ways: the number that scales a rate move into a price move, and the value-weighted *average date* of the bond's cash flows.

Duration, computed

A 3-year bond, 3% annual coupon, yield $y = 2.5\%$:

t_i	cash flow	PV at 2.5%	w_i	$t_i\, w_i$
1	3	2.926	0.0289	0.029
2	3	2.854	0.0282	0.056
3	103	95.558	0.9430	2.829
		$B =$ 101.34	1.0000	$D =$ **2.91**



About 94% of the bond's value is the final payment, so the balance point sits close to year 3: $D = 2.91$ years.

Use it, then check it

Use duration as a quick tangent-line estimate. The curve rises half a percent ($\Delta y = +0.005$):

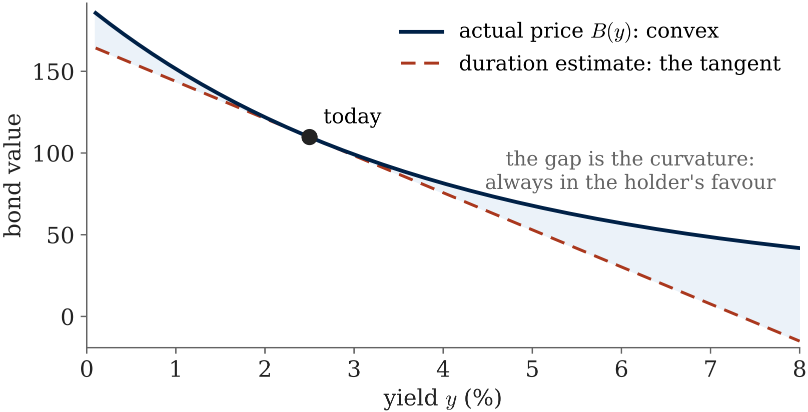
$$\Delta B \approx -\,B\,D\,\Delta y = -\,101.34 \times 2.914 \times 0.005 = -¥1.48.$$

- Exact check: repricing every cash flow at $y = 3.0\%$ gives $B' = 99.87$, a true change of $-¥1.47$.
- The estimate overstates the loss by ¥0.01.

> YOUR TURN — Predict the price change for a *fall* of half a percent. tangent: $+1.48$; exact: $+1.49$

Both errors have the same sign: the true price lands *above* the straight-line estimate on both sides. That is not noise; it is curvature, and it has a name. Vocabulary: 1 basis point (bp) $= 0.01\%$. DV01: the value change per bp, here $\approx ¥0.03$ per ¥100. For a yield quoted with compounding m times a year, use $D^* = D/(1 + y/m)$ (modified duration).

The price-yield relationship is curved



Duration is the tangent line. For small moves it is accurate; for large moves the true price is above the tangent, on both sides. Drawn for a 30-year bond, where the curvature is easy to see; your 3-year bond bends the same way, only less.

Convexity: the second-order correction

Add the next term of the Taylor expansion:

KEY EQUATION · DURATION AND CONVEXITY

$$\frac{\Delta B}{B} \approx -D\,\Delta y + \tfrac{1}{2}\,C\,(\Delta y)^2, \qquad C = \sum_i t_i^2\,w_i$$

For our bond $C = 8.63$, so for $\Delta y = +0.005$:

$$-1.48 + \underbrace{\tfrac{1}{2} \times 8.63 \times (0.005)^2 \times 101.34}_{+0.01} = -1.47.$$

The ¥0.01 residual from two frames ago, reconciled exactly: curvature is systematic and computable.

— *Duration is the slope, convexity the curvature. In Part II the same pair returns for options, renamed delta and gamma.* —

From one bond to a hedged book

A portfolio's duration is the value-weighted average of its bonds' durations, so $\Delta P \approx -P\, D_P\, \Delta y$ applies to the whole book.

portfolio value P	¥80M
portfolio duration D_P	6 years
money duration $P\, D_P$	¥480M·years

Money duration is yuan sensitivity; hedge with the opposite futures sensitivity. The hedging instrument: **CFFEX 10-year T futures**, about ¥1M face per contract, underlying duration $D_F \approx 8$ (contract mechanics: Lecture 6).

$$N = \frac{P\, D_P}{F\, D_F} = \frac{80\text{M} \times 6}{1\text{M} \times 8} = \mathbf{60}\ \text{contracts, short.}$$

— *Lecture 3's procedure, third asset class: match the money-weighted sensitivities, take the offsetting position.* —

Stress test, and what the hedge assumes

A $+50$ basis point move across the whole curve:

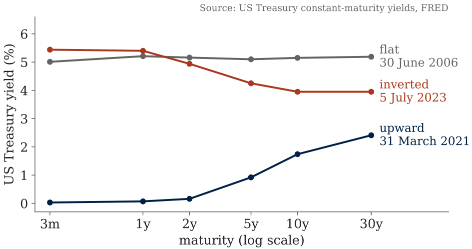
bond portfolio: $-80\text{M} \times 6 \times 0.005$	$-¥2.4\text{M}$
futures: $60 \times 1\text{M} \times 8 \times 0.005$	$+¥2.4\text{M}$
net	$\approx ¥0$

Three assumptions, and what each one can cost you:

- **parallel shifts**: curves also steepen and twist; desks hedge key maturities separately;
- **basis**: the futures track government bonds; a corporate portfolio keeps its credit spread (Lecture 3's basis risk);
- **drift**: durations change as time passes and rates move; re-size the hedge.

And the short futures leg posts variation margin: a bond rally still drains cash (Lecture 2).

Why do curves slope at all?



- **Expectations theory**: forwards equal expected future short rates.
- **Market segmentation**: each maturity has its own lenders and borrowers.
- **Liquidity preference**: you prefer to deposit short but borrow long. Banks push long rates up to balance the two sides, so forwards sit *above* expected rates and curves usually slope upward.

A bank that funds long assets with short money and ignores the mismatch can fail because of it: Silicon Valley Bank, 2023, this week's activity.

Four objects, one curve

zero rate	the price of money for one date; the curve is the full price list
yield	one flat rate summarising one bond, not the whole curve
forward rate	the price of money between two future dates; a tradable breakeven
duration, convexity	slope and curvature of value in the rate

Where this goes next:

- Lecture 5 prices every forward and future with e^{rT}, today's convention.
- Lecture 6 opens the T futures behind our 60-contract hedge: conversion factors, cheapest to deliver.
- Lecture 7 prices swaps as bundles of FRAs, and the par yield returns as the swap rate.