Determination of Forward and Futures Prices

Lecture 5

https://fatih.ai/sjtu/derivatives/

Fatih Kansoy
University of Oxford

Outline

1. Cost-of-Carry Pricing
2. Index and Currency Forwards
3. Storage and Convenience Yield

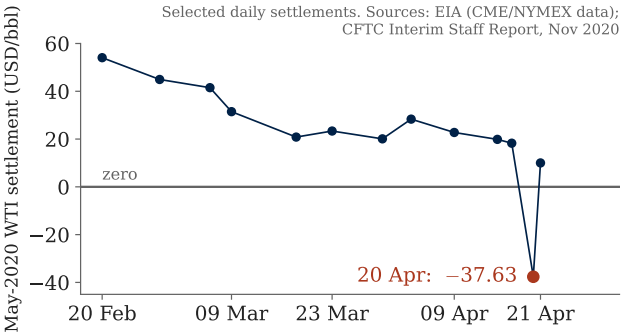
Negative WTI Oil Futures

−$37.63

per barrel: the official settlement of the May-2020 WTI oil future.

Sellers **paid buyers** to take their oil.

CFTC Interim Staff Report, Nov 2020; CME Group.



Two questions, vote on both —

1. How can a price go below zero?
2. Who made money that day?

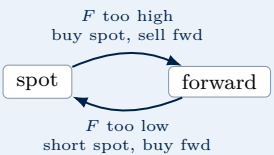
Cost-of-Carry Pricing

Investment Assets and Consumption Assets

HELD TO STORE WEALTH

Investment assets

gold · shares · bonds · currencies

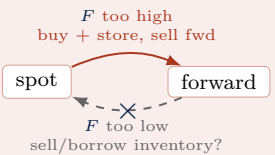


high and low quotes both get corrected

HELD TO USE OR PROCESS

Consumption assets

crude · copper · grain · jet fuel



the low-price correction can fail

An "arm" is one arbitrage direction. Investment assets usually have **two working arms** ⇒ an equation; consumption assets may have only **one working arm** ⇒ an inequality.

— *Oil is a consumption asset, so the low-price arbitrage may not work.* —

Short Selling Mechanics

Our earlier arbitrage tables said "sell or short the asset". The mechanics in one line each:

- **borrow** the asset and **sell** it; later buy it back and return it;
- while it is borrowed, you **owe the lender any income** the asset pays: dividends or coupons (this fact prices the income case below);
- collateral and margin apply, as in our futures mechanics discussion.

In China, A-shares are RMB-denominated stocks listed in Shanghai or Shenzhen. Borrowing these shares for short selling is limited (securities re-lending was suspended in 2024), so investors who want short exposure often use index *futures* instead. This matters later because constrained stock shorting can push index futures below fair value.

Full cash-flow mechanics: appendix.

Forward Price with No Income

For an investment asset with no income and no storage cost, our earlier two-routes argument with continuous compounding gives the gold forward:

$$F_0 = 960 \times e^{0.025\times 1} = ¥984.30.$$

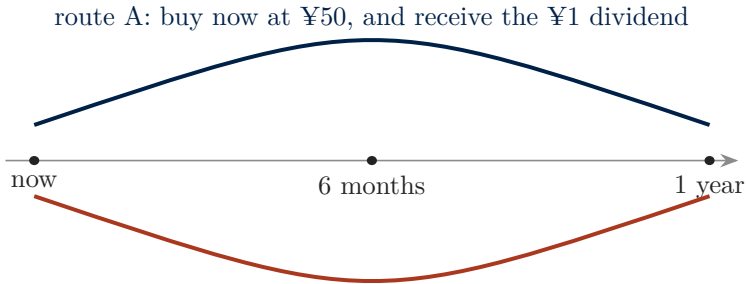
KEY EQUATION · FORWARD ON AN ASSET WITH NO INCOME

$$F_0 = S_0\, e^{rT}$$

- Annual compounding gave ¥984.00; continuous gives ¥984.30. Same result, smoother engine.
- Gold is an investment asset, so both arbitrage arms work: above ¥984.30, borrow, buy spot, sell forward; below, short the gold and buy forward.

Known Income and Forward Prices

A share at ¥50 pays a ¥1 dividend in six months. Two ways to own it in one year:



Route A receives the dividend and route B does not. For the two routes to cost the same, the forward must be cheaper by the value of that dividend.

Known Income Example

Price the one-year forward on the ¥50 share using the previous lecture's curve, $z(0.5) = 2.0\%$ and $z(1) = 2.3\%$:

$$I = 1 \times e^{-0.020 \times 0.5} = 0.990 \qquad \text{discount the dividend at } z(0.5)$$

$$S_0 - I = 50 - 0.990 = 49.010 \qquad \text{today's cost of the share, without its dividend}$$

$$F_0 = 49.010 \times e^{0.023 \times 1} = \mathbf{¥50.15} \qquad \text{carry the package at } z(1)$$

KEY EQUATION · FORWARD ON AN ASSET WITH KNOWN INCOME

$$F_0 = (S_0 - I)\, e^{rT}, \qquad I = \text{present value of the income}$$

Two different rates from one curve: the six-month rate discounts the dividend, the one-year rate carries the position.

Arbitrage for an Overpriced Forward

Suppose the forward is quoted at ¥50.50, above the fair ¥50.15. Sell the overpriced forward and replicate it:

trade leg	cash now	cash at delivery	source of number
borrow ¥50, buy the share	0	repay -51.163	today: $+50 - 50 = 0$; $50e^{0.023} = 51.163$
collect the ¥1 dividend at 6m; reinvest at $f(0.5, 1)$		$+1.013$	$f(0.5, 1) = 2.6\%$; $1e^{0.026 \times 0.5} = 1.013$
short the forward at 50.50	0	deliver, $+50.500$	locked sale price
net	**0**	$+$**¥**0.35	$50.500 + 1.013 - 51.163 = 0.350$

Every rate in the trade, the financing rate and the dividend's reinvestment rate, is read from the curve.

Problem A

PROBLEM A — A share trades at ¥100 and pays a ¥2 dividend in six months. The continuously compounded zero rates are $z(0.5) = 2.0\%$ and $z(1) = 2.3\%$. Price the one-year forward.

ANSWER — $I = 2e^{-0.020 \times 0.5} = 1.980$, so $F_0 = (100 - 1.980)\, e^{0.023} = \mathbf{¥100.30}$.

The same two steps as the ¥50 share: discount the income, subtract, carry. The full step-by-step solution is in the appendix.

Continuous Income Yield

When income accrues continuously in proportion to the price (an index's dividends, a currency's interest), write it as a yield q:

KEY EQUATION · FORWARD ON AN ASSET PAYING A YIELD

$$F_0 = S_0\, e^{(r-q)T}$$

- The same adjustment in a smoother form: the exponent is financing minus what the asset pays the holder.
- Use I for known cash payments on known dates; use q when income is modeled as a continuous percentage yield.

If holding the asset earns income before delivery, subtract that income when pricing the forward.

Value of an Existing Forward Contract

Old contract: long 100,000 tonnes at $K = ¥6{,}300$. Six months remain. Current same-delivery forward: $F_0 = ¥6{,}500$. The old long buys below today's forward price.

position	cash at delivery	source of number
old long forward	$-6{,}300$	old delivery price K
new short forward	$+6{,}500$	current forward F_0; zero cost today
locked cash flow	$+¥200$/t	$F_0 - K = 6{,}500 - 6{,}300$
value today	$+¥198.01$/t	$200e^{-0.020\times 0.5}$

KEY EQUATION · VALUE OF AN EXISTING LONG FORWARD

$$f = (F_0 - K)e^{-rT}, \qquad T = \text{time remaining}$$

Full position value: $198.01 \times 100{,}000 = +¥19.8$M to the long, $-¥19.8$M to the short.

Index and Currency Forwards

CSI 300 Index Futures Fair Value

A stock index is a portfolio of shares paying dividends: exactly the yield case, with S_0 the index level and q its dividend yield.

index level, S_0		4,000
one-year rate, r	(from the curve)	2.3%
dividend yield, q	(illustrative)	2.0%
$F_0 = 4{,}000\, e^{(0.023-0.020)\times 1}$		**4,012**

The two carry legs nearly cancel: financing costs 2.3%, the dividends pay back 2.0%, so the futures sit only twelve points above the index.

With known interest rates, the same no-arbitrage price applies to forwards and futures; proof in appendix A6.

Index Arbitrage Constraints

The 4,012 number is a **fair futures price today**, not a forecast. It is the no-arbitrage benchmark: spot 4,000 carried by net carry $r - q = 0.3\%$.

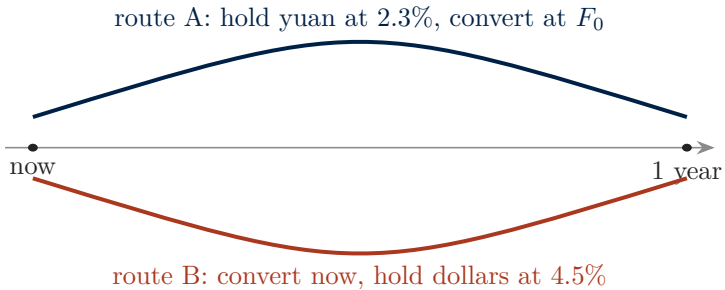
- **Futures above 4,012:** buy the stock basket and short the futures. This high-price correction is the standard cash-and-carry.
- **Futures below 4,012:** buy futures and short the stock basket. This low-price correction needs share borrowing.
- In China, A-share short selling is limited, so investors wanting short exposure often use index futures. The low-price correction is weak.
- Mainland index futures have traded **below fair value** for long stretches: a discount, largest in the small-cap contracts.

The discount is not free profit. It is the price of the short-sale constraint and a real running cost of short hedges.

If the low-price correction is hard to trade, futures can remain below fair value.

Currency Forwards and Foreign Interest

A dollar invoice of **$100M** is due in one year; the spot rate is $S_0 = 7.20$ CNY/USD. Dollars held in a US account earn $r_{\$} = 4.5\%$; yuan finance at $r_{¥} = 2.3\%$. The foreign interest rate is the yield q.



Both routes turn today's yuan into dollars at the same date. No-arbitrage forces them to deliver the same amount: covered interest parity.

Covered Interest Parity Example

Start with 7.20 yuan, worth \$1 today. Let F_0 be the one-year forward exchange rate in CNY per USD. Equate the two routes' dollars at one year:

$$\frac{7.20\, e^{0.023}}{F_0} = 1 \times e^{0.045} \qquad \text{route A dollars} = \text{route B dollars}$$

$$F_0 = 7.20\, e^{0.023-0.045} \qquad \text{solve for } F_0\text{: exponents subtract}$$

$$= 7.20\, e^{-0.022} = \mathbf{7.043} \qquad \text{the one-year forward, CNY/USD}$$

KEY EQUATION · COVERED INTEREST PARITY

$$F_0 = S_0\, e^{(r_{\text{CNY}} - r_{\text{USD}})T}$$

Here both S_0 and F_0 are exchange rates quoted as **CNY per USD**. The domestic rate is the CNY rate; the foreign rate is the USD rate.

Currency Forward Interpretation

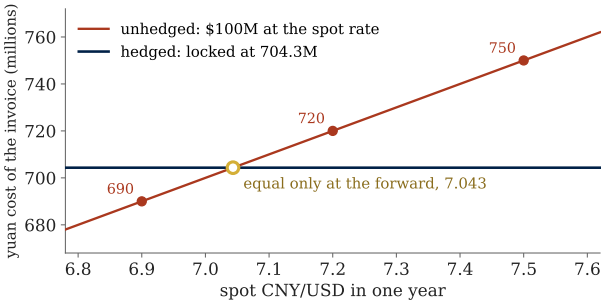
$F_0 = 7.043$: the forward dollar is cheaper than the spot dollar (7.20). Three readings, two of them wrong:

- *"the market expects a stronger yuan"*: no; forwards are breakevens, not forecasts;
- *"locking at 7.04 beats 7.20, so it is free money"*: no; the discount exactly offsets the higher dollar interest you give up;
- **parity**: the higher-yielding currency must trade at a forward discount, or one of the two routes is an arbitrage.

A currency forward is a break-even financing rate, not an exchange-rate forecast.

Currency Forward Hedge

Buy \$100M forward at 7.043 CNY/USD. Whatever the future spot rate is, the yuan cost is locked at $100\text{M} \times 7.043 = ¥704.3\text{M}$:



future spot	forward gain/loss	**net cost**
6.90	−¥14.3M	**¥704.3M**
7.20	+¥15.7M	**¥704.3M**
7.50	+¥45.7M	**¥704.3M**

The forward gain or loss offsets the spot invoice cost, so every row ends at ¥704.3M.

PROBLEM B — The invoice is due in *six* months instead. Find the forward. Answer: $7.20\,e^{-0.022\times 0.5} = 7.121$.

Storage and Convenience Yield

Storage Costs in Forward Prices

Holding a physical commodity costs money: tanks, yards, insurance. Storage is negative income; as a continuous rate u it joins the financing leg. Gold with storage cost $u = 0.3\%$:

$$F_0 = 960 \times e^{(0.025+0.003)\times 1} = ¥987.26 \quad \text{(was 984.30)}.$$

KEY EQUATION · FORWARD ON AN ASSET WITH STORAGE COSTS

$$F_0 = S_0\, e^{(r+u)T}$$

Storage is a holding cost, so it raises the forward price through the $+u$ term.

Consumption Asset Pricing Inequality

Apply both arbitrage arms to crude oil:

- **Futures too high?** Borrow, buy crude, store it, short the futures. Anyone with a tank can do this, so the ceiling holds.
- **Futures too low?** The trade requires selling crude held today and buying it back forward. But the holders are refiners, and selling the feedstock means halting production. They will not supply that arm.

The floor argument fails, and only the ceiling survives:

KEY EQUATION · CONSUMPTION ASSETS: AN INEQUALITY, NOT AN EQUATION

$$F_0 \;\leq\; S_0\, e^{(r+u)T}$$

Convenience Yield

A refinery may keep crude even when the forward price is cheaper than spot. The reason is not cash income; it is operational value.

- physical inventory keeps production running;
- it protects against delivery delays, local shortages, and price spikes;
- a forward contract gives crude later, but inventory gives crude now.

DEFINITION — The **convenience yield** y is the non-cash benefit of holding the physical asset now.

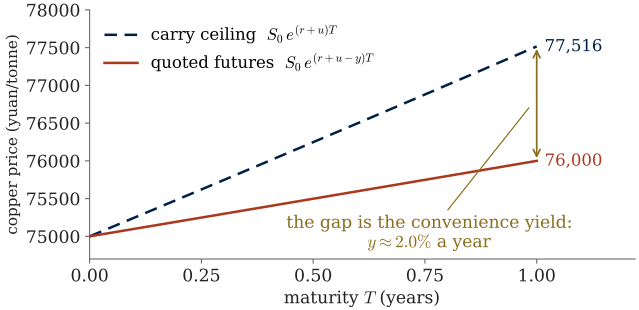
It enters the carry formula like income because only the spot holder receives this benefit:

KEY EQUATION · CONSUMPTION ASSETS, WITH THE GAP NAMED

$$F_0 = S_0\, e^{(r+u-y)T}$$

High inventories $\Rightarrow$ low y; tight inventories $\Rightarrow$ high y.

Implied Convenience Yield



no-convenience carry ceiling	¥75,000$e^{0.033}$ = ¥77,516
observed one-year futures quote	¥76,000

A futures price below the carry ceiling means the market values having the metal available now.

Implied Convenience Yield Calculation

Use the observed copper futures price in $F_0 = S_0 e^{(r+u-y)T}$ and solve for y.

$$76{,}000 = 75{,}000\, e^{(0.023+0.010-y)\times 1} \qquad \text{insert the futures quote}$$

$$0.033 - y = \ln \tfrac{76{,}000}{75{,}000} \qquad \text{take logs}$$

$$y = 0.033 - 0.0132 = \mathbf{2.0}\% \qquad \text{the implied convenience yield}$$

Two per cent per year is the market's implied value of having copper now rather than later.

PROBLEM C — Futures print ¥75,500 instead: re-imply y. Answer: $y = 2.64\%$ (A12).

— *Lower futures relative to spot imply a higher convenience yield.* —

Cost-of-Carry Formula

PUSHES THE FORWARD UP

Holding it costs

$+r$ financing	every asset
$+u$ storage	physical assets

costs are added to carry

PUSHES THE FORWARD DOWN

Holding it pays

$-q$ income yield	shares, index
$-r_f$ foreign rate	currency case
$-y$ convenience	commodities

benefits are subtracted

NET CARRY COMPRESSED INTO ONE EXPONENT

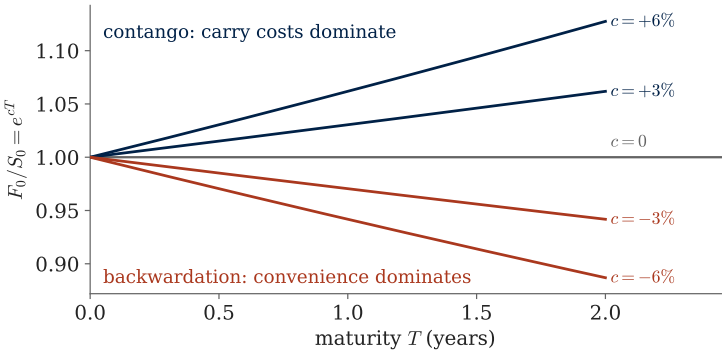
$$F_0 = S_0\, e^{(\,r+u\,-\,q\,-\,y\,)T}$$

Dated cash income uses the same subtraction before carrying: $F_0 = (S_0 - I)e^{rT}$.

Signs in the exponent: costs are +, benefits are −, missing terms are zero.

Contango and Backwardation

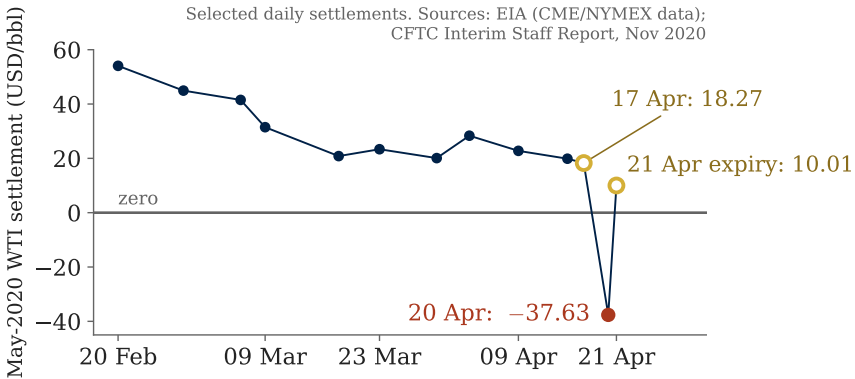
The net carry $c = r + u - q - y$ decides which one you get:



Contango (futures above spot): storage and financing dominate.

Backwardation (futures below spot): convenience dominates, because availability is valued now.

April 2020 WTI Storage Constraint



The May WTI contract went negative just before expiry because deliverable storage became scarce.

WTI Storage Constraint Explanation

The normal cash-and-carry trade needs a physical place to put the oil:

buy May futures / take delivery → store one month → sell June futures.

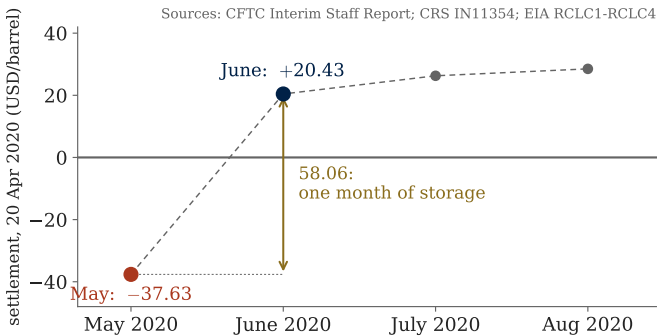
If that trade is possible, it limits how wide the May–June spread can become. In April 2020 the storage step failed:

- Cushing, Oklahoma (the delivery point) holds about 76 million barrels; by 17 April it was about **76% full**, and the remaining space was already leased or committed;
- a long May futures position was close to becoming physical delivery;
- a trader without storage had to sell to someone with a tank, even at a negative price.

— *Negative WTI was not "oil has no value"; it was "taking delivery here and now is costly."* —

May-June WTI Spread

The same afternoon, the June contract settled at +$20.43. Two prices, one month apart, $58.06 apart: the market's price for one month of storage.



Whoever could supply storage collected it: within a week, oil held in tankers at sea hit a record of about 160 million barrels. Buy May, sell June, charter the ship: paid the scarcity price of a warehouse.

Summary

- A forward price is **the spot price, carried**: financing and storage add, income and convenience subtract, $F_0 = S_0 e^{(r+u-q-y)T}$.
- **Inequalities carry information.** Where one arbitrage arm is constrained, the surviving gap is a price you can read: the index discount, the convenience yield.
- **The assumptions are themselves prices.** Storage, shorting, and availability each turned out to be a market, and April 2020 is what happens when one of those markets runs out.

The course so far: derivatives transfer risk; clearing concentrates it; hedging exchanges it; every date has a price; a forward price is the spot price, carried.

Appendix: details behind the main slides

A1. Why Short Sellers Pay Dividends

A short seller borrows someone else's share. That is why the short must pass through any dividend paid while the position is open.

cash flow	long shareholder	short seller
April: 500 shares at \$120	−60,000	+60,000
May: \$1 dividend per share	+500	−500
July: buy back / sell at \$100	+50,000	−50,000
net before margin and fees	−\$9,500	+\$9,500

This is the cash-flow reason income reduces the fair forward price: the asset holder receives income, while the short seller owes it to the lender.

A2. Low Forward Price Arbitrage

The ¥50 share's fair one-year forward is ¥50.15. If the quote is ¥49.90, the forward is too low, so use the low-price correction: short the share and buy the forward.

trade leg	cash now	cash at delivery
short share at ¥50 and invest proceeds	0	$+51.163$
pay ¥1 dividend at 6m; carry it to T	0	-1.013
buy forward at ¥49.90; return the share	0	-49.900
locked profit	**0**	$+$**¥**0.250

The numbers use the same curve as the main example: $50e^{0.023} = 51.163$ and $1e^{0.026\times 0.5} = 1.013$.

A3. Known Income Formula Proof

For known cash income with present value I, the fair forward is $F_0 = (S_0 - I)e^{rT}$.

if forward is too high	if forward is too low
borrow S_0, buy the asset	short the asset, invest S_0
receive income worth I today	owe income worth I today
loan effectively grows from $S_0 - I$	investment effectively grows from $S_0 - I$
deliver into forward	buy through forward and return asset
profit $F_0 - (S_0 - I)e^{rT}$	profit $(S_0 - I)e^{rT} - F_0$

If both trades are feasible, neither profit can survive. That pins down the equality.

A4. Continuous Yield Formula Proof

When income is a continuous proportional yield q, use a units argument:

buy e^{-qT} units today, cost $S_0 e^{-qT}$	less than one unit
reinvest the income in the asset	units grow at rate q
$e^{-qT} \times e^{qT} = 1$	exactly one unit by delivery
$F_0 = S_0 e^{-qT} e^{rT} = S_0 e^{(r-q)T}$	carry the financing cost

The same idea prices an index with dividend yield and a currency with foreign interest rate.

A5. Existing Forward Contract Value

A forward price is set at inception. Later, the contract has value because the current fair delivery price may have moved.

old long forward struck at K	already locked in
new short forward at today's F_0	costs zero today
cash at T : $(S_T - K) + (F_0 - S_T) = F_0 - K$	spot risk cancels
$f = (F_0 - K)e^{-rT}$	discount the certain cash flow

At inception $K = F_0$, so $f = 0$. That means the contract starts with zero value, not zero notional.

A6. Why Futures Can Use the Forward Formula

A forward settles once; a futures contract settles every day. With known interest rates, daily settlement does not change the fair price.

- Each futures gain or loss can be invested or financed at known rates until delivery.
- By choosing the futures position size to account for those known discount factors, the daily cash flows can be made equivalent to one forward payoff.
- Therefore the CSI 300 futures fair-value slide can use the forward formula when rates are treated as known.

If interest rates are random, the two prices can differ slightly because futures gains and losses arrive on different days.

A7. Covered Interest Parity Arbitrage

Fair one-year forward: $F^* = 7.043$ CNY/USD. If the market quote moves away from this, one of the two financing routes is cheaper.

quote 7.10: USD expensive	**quote 6.98: USD cheap**
borrow ¥7.20, convert to $1	borrow $1, convert to ¥7.20
invest dollars: $1.046 at T	invest yuan: ¥7.368 at T
sell $1.046 forward at 7.10: ¥7.427	buy $1.046 forward at 6.98: costs ¥7.301
repay yuan loan: ¥7.368	repay dollar loan: $1.046
profit ¥0.059 per dollar	**profit ¥0.066 per dollar**

Small per-dollar gaps become large because FX forwards trade on large notionals.

A8. CNY, CNH, and Parity Frictions

Covered interest parity is strongest when money can move freely between the two currencies. The renminbi case is useful because that movement can be restricted.

- **CNY** is the onshore renminbi market; **CNH** is the offshore renminbi market.
- Their spot and forward prices can differ because cross-border funding and conversion are not always frictionless.
- The gap is not automatically an arbitrage error; it can be the price of the restricted funding route.

Same lesson as index futures: when the correcting trade is constrained, the formula becomes a benchmark and the gap becomes information.

A9. China Index Futures Discounts

The fair-value benchmark is $F^* = S_0 e^{(r-q)T}$. A discount means the traded futures price is below that benchmark.

- The correcting trade is: buy the cheap index future and short the stock basket.
- In China, A-share borrowing has often been constrained; the 2024 suspension of securities re-lending is one example of tighter share availability.
- When direct stock shorting is hard, short exposure can crowd into index futures, pushing futures below fair value.

The formula did not fail. It tells us which arbitrage arm is blocked.

A10. Problem A Solution

Inputs: $S_0 = ¥100$, dividend $= ¥2$ in six months, $z(0.5) = 2.0\%$, $z(1) = 2.3\%$.

$$I = 2e^{-0.020\times 0.5} = 1.980 \qquad \text{discount the known dividend}$$

$$S_0 - I = 100 - 1.980 = 98.020 \qquad \text{remove income received before delivery}$$

$$F_0 = 98.020e^{0.023\times 1} \qquad \text{carry to one year}$$

$$= \mathbf{¥100.30}$$

The dividend uses the six-month zero rate because the cash is paid in six months. The remaining carry uses the one-year zero rate because delivery is in one year.

A11. Problem B Solution

Inputs: six-month dollar invoice, $S_0 = 7.20$ CNY/USD, $r_{\text{CNY}} = 2.3\%$, $r_{\text{USD}} = 4.5\%$.

$$F_0 = 7.20e^{(0.023-0.045)\times 0.5} \qquad \text{CNY per USD quotation}$$

$$= 7.20e^{-0.011} \qquad \text{half-year horizon}$$

$$= \mathbf{7.121} \qquad \text{six-month forward rate}$$

The forward is closer to spot than the one-year 7.043 rate because the interest-rate gap has only six months to act.

A12. Problem C Solution

Copper futures quote: ¥75,500. Spot: ¥75,000. Net cost before convenience is $r + u = 3.3\%$.

$$75{,}500 = 75{,}000 e^{(0.033 - y) \times 1} \qquad \text{observed futures quote}$$

$$0.033 - y = \ln\left(\frac{75{,}500}{75{,}000}\right) = 0.00664 \qquad \text{take logs}$$

$$y = 0.033 - 0.00664 = \mathbf{2.64}\%$$

A lower futures price relative to spot means a higher implied convenience yield: the market values having copper available now.

A13. Forward Price Is Not a Forecast

F_0 is fixed by today's spot price and carry costs. It is not automatically the market's best forecast of the future spot price.

comparison	what it tells us
market quote vs fair F_0	arbitrage signal, unless a trading arm is constrained
fair F_0 vs spot S_0	financing, income, storage, convenience
fair F_0 vs expected S_T	risk premia and beliefs, not pure arbitrage

Teaching rule: use carry to price the forward; use separate economic judgment to discuss forecasts.

A14. Delivery Details Matter

The cost-of-carry formula hides a physical contract underneath. For deliverable futures, the contract specifies what can be delivered.

- **Grade**: acceptable quality and price adjustments.
- **Location**: approved warehouse, port, or delivery point.
- **Timing**: delivery window inside the delivery month.

These details explain why the WTI story was about *Cushing storage*, not just about the global price of oil. Delivery terms decide which arbitrage trades are actually feasible.