Financial Econometrics
Shangai Jiao Tong University Summer School 2025-2026
Problem Set - Lecture 8

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Reference: Stock, J.H. and Watson, M.W., Introduction to Econometrics (4th Edition, Global Edition)

Chapter 16: Estimation of Dynamic Causal Effects

Question 1. Consider the regression model $Y_t = \beta_0 + \beta_1 X_t + u_t$, where u_t follows the stationary AR(1) model $u_t = \phi_1 u_{t-1} + \tilde{u}_t$ with $\tilde{u}_t$ i.i.d. with mean 0 and variance $\sigma^2_{\tilde{u}}$ and $|\phi_1| < 1$; the regressor X_t follows the stationary AR(1) model $X_t = \gamma_1 X_{t-1} + e_t$ with e_t i.i.d. with mean 0 and variance σ^2_e and $|\gamma_1| < 1$; and e_t is independent of $\tilde{u}_i$ for all t and i.

(a) Show that $\text{var}(u_t) = \sigma^2_{\tilde{u}}/(1 - \phi_1^2)$ and $\text{var}(X_t) = \sigma^2_e/(1 - \gamma_1^2)$.

(b) Show that $\text{cov}(u_t, u_{t-j}) = \phi_1^j \text{var}(u_t)$ and $\text{cov}(X_t, X_{t-j}) = \gamma_1^j \text{var}(X_t)$.

(c) Show that $\text{corr}(u_t, u_{t-j}) = \phi_1^j$ and $\text{corr}(X_t, X_{t-j}) = \gamma_1^j$.

(d) Consider the terms σ^2_v and f_T in Equation (16.14).

   (i) Show that $\sigma^2_v = \sigma^2_X \sigma^2_u$, where σ^2_X is the variance of X and σ^2_u is the variance of u.

   (ii) Derive an expression for f_∞.

Question 2. Suppose $Y_t = \beta_0 + u_t$, where u_t follows a stationary AR(1): $u_t = \phi_1 u_{t-1} + \tilde{u}_t$ with $\tilde{u}_t$ i.i.d. with mean 0 and variance $\sigma^2_{\tilde{u}}$ and $|\phi_1| < 1$.

(a) Show that $\beta_0 = \mu_Y = E(Y_t)$.

(b) Let $\bar{Y}_{1:T} = \frac{1}{T}\sum_{t=1}^{T} Y_t$ denote the sample mean. Show that the OLS estimator of β_0 is $\hat{\beta}_0 = \bar{Y}_{1:T}$.

(c) Show that $\text{var}[\sqrt{T}(\bar{Y}_{1:T} - \mu_Y)] \to \sigma^2_{\tilde{u}}/(1-\phi_1)^2$.

(d) Assume $\bar{Y}_{1:T}$ is approximately normally distributed with mean μ_Y and variance $\sigma^2_{\tilde{u}}/[T(1-\phi_1)^2]$. Suppose $T = 200$, $\sigma^2_{\tilde{u}} = 7.9$, $\phi_1 = 0.3$, and $\bar{Y}_{1:T} = 2.8$. Construct a 95% confidence interval for μ_Y.

Chapter 17: Additional Topics in Time Series

Question 3. According to the **expectations theory** of the term structure of interest rates, the k-period interest rate is the average of expected future one-period rates:

$$R_{kt} = \frac{1}{k}\sum_{i=0}^{k-1} E_t(R_{1,t+i}) + e_t$$

where R_{kt} is the k-period rate, R_{1t} is the one-period rate, and e_t is a stationary term premium.

(a) Suppose R_{1t} follows a random walk: $R_{1t} = R_{1,t-1} + u_t$. Show that $R_{kt} = R_{1t} + e_t$.

(b) Under the assumptions in (a), are R_{kt} and R_{1t} cointegrated? If so, what is the cointegrating coefficient?

(c) Suppose $\Delta R_{1t} = \phi_1 \Delta R_{1,t-1} + u_t$ where $|\phi_1| < 1$. Are R_{kt} and R_{1t} cointegrated?

(d) Suppose $R_{1t} = 0.5R_{1,t-1} + u_t$. Are R_{kt} and R_{1t} cointegrated?

Question 4. Consider the regression:

$$Y_t = 2.0 + 3.6X_{t+1} - 1.2X_t - 0.3X_{t-1} + u_t$$

(a) Rewrite this in the form $Y_t = \beta_0 + \beta X_t + \delta_{-1}\Delta X_{t+1} + \delta_0 \Delta X_t + u_t$ and identify β, δ_{-1}, and δ_0.

(b) Cointegration between Y_t and X_t requires that X_t is $I(1)$ and u_t is $I(0)$. For each case below, are Y_t and X_t cointegrated?

(i) $X_t \sim I(0)$, $u_t \sim I(0)$

(ii) $X_t \sim I(1)$, $u_t \sim I(1)$

(iii) $X_t \sim I(1)$, $u_t \sim I(0)$

Question 5. Consider the GARCH(1,1) model:

$$u_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.}(0,1)$$

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2$$

where $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$.

(a) Show that the unconditional variance of u_t is $\text{var}(u_t) = \dfrac{\omega}{1-\alpha-\beta}$.

(b) Derive $E(\sigma_t^2)$.

(c) What restriction on the parameters is needed for $\text{var}(u_t)$ to be positive and finite?

(d) For the ARCH(1) model ($\beta = 0$), derive the unconditional variance.

(e) What is the stationarity condition for ARCH(1)?