Financial Econometrics
Shangai Jiao Tong University
Summer School – 2025-2026
Problem Set - Lecture 8
With Solutions

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Reference: Stock, J.H. and Watson, M.W., Introduction to Econometrics (4th Edition, Global Edition)

Chapter 16: Estimation of Dynamic Causal Effects

Question 1. Consider the regression model $Y_t = \beta_0 + \beta_1 X_t + u_t$, where u_t follows the stationary AR(1) model $u_t = \phi_1 u_{t-1} + \tilde{u}_t$ with $\tilde{u}_t$ i.i.d. with mean 0 and variance $\sigma^2_{\tilde{u}}$ and $|\phi_1| < 1$; the regressor X_t follows the stationary AR(1) model $X_t = \gamma_1 X_{t-1} + e_t$ with e_t i.i.d. with mean 0 and variance σ^2_e and $|\gamma_1| < 1$; and e_t is independent of $\tilde{u}_i$ for all t and i.

(a) Show that $\text{var}(u_t) = \sigma^2_{\tilde{u}}/(1 - \phi_1^2)$ and $\text{var}(X_t) = \sigma^2_e/(1 - \gamma_1^2)$.

(b) Show that $\text{cov}(u_t, u_{t-j}) = \phi_1^j \text{var}(u_t)$ and $\text{cov}(X_t, X_{t-j}) = \gamma_1^j \text{var}(X_t)$.

(c) Show that $\text{corr}(u_t, u_{t-j}) = \phi_1^j$ and $\text{corr}(X_t, X_{t-j}) = \gamma_1^j$.

(d) Consider the terms σ^2_v and f_T in Equation (16.14).

  (i) Show that $\sigma^2_v = \sigma^2_X \sigma^2_u$, where σ^2_X is the variance of X and σ^2_u is the variance of u.

  (ii) Derive an expression for f_∞.

Solution:

(a) **Variance of AR(1) process:**

For $u_t = \phi_1 u_{t-1} + \tilde{u}_t$:

$$\begin{aligned}\text{var}(u_t) &= \text{var}(\phi_1 u_{t-1} + \tilde{u}_t) \\ &= \phi_1^2 \text{var}(u_{t-1}) + \text{var}(\tilde{u}_t) + 2\phi_1 \text{cov}(u_{t-1}, \tilde{u}_t)\end{aligned}$$

Since $\tilde{u}_t$ is independent of past values, $\text{cov}(u_{t-1}, \tilde{u}_t) = 0$.

By stationarity, $\text{var}(u_t) = \text{var}(u_{t-1}) = \sigma_u^2$.

Therefore:

$$\sigma_u^2 = \phi_1^2 \sigma_u^2 + \sigma_{\tilde{u}}^2 \implies \sigma_u^2(1 - \phi_1^2) = \sigma_{\tilde{u}}^2$$

$$\boxed{\text{var}(u_t) = \frac{\sigma_{\tilde{u}}^2}{1 - \phi_1^2}}$$

By identical reasoning for X_t:

$$\boxed{\text{var}(X_t) = \frac{\sigma_e^2}{1 - \gamma_1^2}}$$

(b) **Autocovariances:**

For $j = 1$: Multiply both sides of $u_t = \phi_1 u_{t-1} + \tilde{u}_t$ by u_{t-1} and take expectations:

$$\begin{aligned}E(u_t u_{t-1}) &= \phi_1 E(u_{t-1}^2) + E(\tilde{u}_t u_{t-1}) \\ \text{cov}(u_t, u_{t-1}) &= \phi_1 \text{var}(u_{t-1}) + 0 = \phi_1 \sigma_u^2\end{aligned}$$

For $j = 2$: $u_t = \phi_1^2 u_{t-2} + \phi_1 \tilde{u}_{t-1} + \tilde{u}_t$

$$\text{cov}(u_t, u_{t-2}) = \phi_1^2 \text{var}(u_{t-2}) = \phi_1^2 \sigma_u^2$$

By induction:

$$\boxed{\text{cov}(u_t, u_{t-j}) = \phi_1^j \text{var}(u_t)}$$

Similarly:

$$\boxed{\text{cov}(X_t, X_{t-j}) = \gamma_1^j \text{var}(X_t)}$$

(c) **Autocorrelations:**

$$\text{corr}(u_t, u_{t-j}) = \frac{\text{cov}(u_t, u_{t-j})}{\sqrt{\text{var}(u_t)}\sqrt{\text{var}(u_{t-j})}} = \frac{\phi_1^j \sigma_u^2}{\sigma_u^2} = \boxed{\phi_1^j}$$

Similarly: $\operatorname{corr}(X_t, X_{t-j}) = \boxed{\gamma_1^j}$

Key insight: Autocorrelations decay geometrically for AR(1) processes.

(d) **Variance components in the HAC formula:**

Define $v_t = (X_t - \mu_X)u_t$.

(i) Since X_t and u_t are independent (as $e_t \perp \tilde{u}_i$ for all t, i):

$$\begin{aligned}
\sigma_v^2 &= E[v_t^2] = E[(X_t - \mu_X)^2 u_t^2] \\
&= E[(X_t - \mu_X)^2] \cdot E[u_t^2] \quad \text{(independence)} \\
&= \sigma_X^2 \cdot \sigma_u^2
\end{aligned}$$

$$\boxed{\sigma_v^2 = \sigma_X^2 \sigma_u^2}$$

(ii) The long-run variance is:

$$f_\infty = \sum_{j=-\infty}^{\infty} \operatorname{cov}(v_t, v_{t-j})$$

Since $\operatorname{cov}(v_t, v_{t-j}) = E[(X_t - \mu_X)(X_{t-j} - \mu_X)] \cdot E[u_t u_{t-j}]$:

$$\operatorname{cov}(v_t, v_{t-j}) = \gamma_1^{|j|}\sigma_X^2 \cdot \phi_1^{|j|}\sigma_u^2 = (\gamma_1\phi_1)^{|j|}\sigma_X^2\sigma_u^2$$

Therefore:

$$\begin{aligned}
f_\infty &= \sigma_X^2\sigma_u^2 \sum_{j=-\infty}^{\infty} (\gamma_1\phi_1)^{|j|} \\
&= \sigma_X^2\sigma_u^2 \left[1 + 2\sum_{j=1}^{\infty}(\gamma_1\phi_1)^j\right] \\
&= \sigma_X^2\sigma_u^2 \cdot \frac{1+\gamma_1\phi_1}{1-\gamma_1\phi_1}
\end{aligned}$$

$$\boxed{f_\infty = \sigma_X^2\sigma_u^2 \cdot \frac{1+\gamma_1\phi_1}{1-\gamma_1\phi_1}}$$

Question 2. Suppose $Y_t = \beta_0 + u_t$, where u_t follows a stationary AR(1): $u_t = \phi_1 u_{t-1} + \tilde{u}_t$ with $\tilde{u}_t$ i.i.d. with mean 0 and variance $\sigma_{\tilde{u}}^2$ and $|\phi_1| < 1$.

(a) Show that $\beta_0 = \mu_Y = E(Y_t)$.

(b) Let $\bar{Y}_{1:T} = \frac{1}{T}\sum_{t=1}^{T} Y_t$ denote the sample mean. Show that the OLS estimator of β_0 is $\hat{\beta}_0 = \bar{Y}_{1:T}$.

(c) Show that $\text{var}[\sqrt{T}(\bar{Y}_{1:T} - \mu_Y)] \to \sigma^2_{\tilde{u}}/(1-\phi_1)^2$.

(d) Assume $\bar{Y}_{1:T}$ is approximately normally distributed with mean μ_Y and variance $\sigma^2_{\tilde{u}}/[T(1-\phi_1)^2]$. Suppose $T = 200$, $\sigma^2_{\tilde{u}} = 7.9$, $\phi_1 = 0.3$, and $\bar{Y}_{1:T} = 2.8$. Construct a 95% confidence interval for μ_Y.

Solution:

(a) **Expected value:**

Taking expectations of $Y_t = \beta_0 + u_t$:

$$E(Y_t) = \beta_0 + E(u_t)$$

Since u_t is a stationary AR(1) with $E(\tilde{u}_t) = 0$:

$$E(u_t) = \phi_1 E(u_{t-1}) + E(\tilde{u}_t) = \phi_1 E(u_t) + 0$$

$$E(u_t)(1-\phi_1) = 0 \implies E(u_t) = 0$$

Therefore:

$$\boxed{\beta_0 = \mu_Y = E(Y_t)}$$

(b) **OLS estimator:**

The model $Y_t = \beta_0 + u_t$ is a regression with only a constant. OLS minimizes:

$$\sum_{t=1}^{T}(Y_t - b_0)^2$$

FOC: $-2\sum_{t=1}^{T}(Y_t - \hat{\beta}_0) = 0$

Solving: $T\hat{\beta}_0 = \sum_{t=1}^{T} Y_t$

$$\boxed{\hat{\beta}_0 = \bar{Y}_{1:T} = \frac{1}{T}\sum_{t=1}^{T} Y_t}$$

(c) **Asymptotic variance:**

From the AR(1) structure:

$$\text{var}(u_t) = \frac{\sigma^2_{\tilde{u}}}{1-\phi_1^2}, \quad \text{cov}(u_t, u_{t-j}) = \phi_1^j \cdot \text{var}(u_t)$$

The variance of $\bar{Y}_{1:T} - \mu_Y = \bar{u}_{1:T}$ is:

$$\text{var}(\bar{u}_{1:T}) = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\text{cov}(u_t, u_s)$$

For large T:

$$T \cdot \text{var}(\bar{u}_{1:T}) \to \sum_{j=-\infty}^{\infty}\text{cov}(u_t, u_{t-j}) = \text{var}(u_t) \cdot \frac{1+\phi_1}{1-\phi_1}$$

Substituting $\text{var}(u_t) = \sigma_{\tilde{u}}^2/(1-\phi_1^2) = \sigma_{\tilde{u}}^2/[(1-\phi_1)(1+\phi_1)]$:

$$T \cdot \text{var}(\bar{u}_{1:T}) \to \frac{\sigma_{\tilde{u}}^2}{(1-\phi_1)(1+\phi_1)} \cdot \frac{1+\phi_1}{1-\phi_1} = \frac{\sigma_{\tilde{u}}^2}{(1-\phi_1)^2}$$

$$\boxed{\text{var}[\sqrt{T}(\bar{Y}_{1:T} - \mu_Y)] \to \frac{\sigma_{\tilde{u}}^2}{(1-\phi_1)^2}}$$

(d) **95% confidence interval:**

Given: $T = 200$, $\sigma_{\tilde{u}}^2 = 7.9$, $\phi_1 = 0.3$, $\bar{Y}_{1:T} = 2.8$

Standard error:

$$SE(\bar{Y}) = \sqrt{\frac{\sigma_{\tilde{u}}^2}{T(1-\phi_1)^2}} = \sqrt{\frac{7.9}{200 \times (0.7)^2}} = \sqrt{\frac{7.9}{98}} = \sqrt{0.0806} = 0.284$$

95% CI:

$$\bar{Y} \pm 1.96 \times SE = 2.8 \pm 1.96 \times 0.284 = 2.8 \pm 0.557$$

$$\boxed{[2.24, 3.36]}$$

Chapter 17: Additional Topics in Time Series

Question 3. According to the **expectations theory** of the term structure of interest rates, the k-period interest rate is the average of expected future one-period rates:

$$R_{kt} = \frac{1}{k}\sum_{i=0}^{k-1} E_t(R_{1,t+i}) + e_t$$

where R_{kt} is the k-period rate, R_{1t} is the one-period rate, and e_t is a stationary term premium.

(a) Suppose R_{1t} follows a random walk: $R_{1t} = R_{1,t-1} + u_t$. Show that $R_{kt} = R_{1t} + e_t$.

(b) Under the assumptions in (a), are R_{kt} and R_{1t} cointegrated? If so, what is the cointegrating coefficient?

(c) Suppose $\Delta R_{1t} = \phi_1 \Delta R_{1,t-1} + u_t$ where $|\phi_1| < 1$. Are R_{kt} and R_{1t} cointegrated?

(d) Suppose $R_{1t} = 0.5R_{1,t-1} + u_t$. Are R_{kt} and R_{1t} cointegrated?

Solution:

(a) When R_{1t} follows a random walk $R_{1t} = R_{1,t-1} + u_t$, the optimal i-period ahead forecast is simply the current value:

$$E_t(R_{1,t+i}) = R_{1t} \quad \text{for all } i \geq 0$$

Substituting into the expectations theory formula:

$$R_{kt} = \frac{1}{k}\sum_{i=0}^{k-1} E_t(R_{1,t+i}) + e_t = \frac{1}{k}\sum_{i=0}^{k-1} R_{1t} + e_t = \frac{1}{k} \cdot k \cdot R_{1t} + e_t = \boxed{R_{1t} + e_t}$$

(b) Yes, R_{kt} and R_{1t} are cointegrated.

Verification of cointegration conditions:

- R_{1t} follows a random walk $\Rightarrow R_{1t} \sim I(1)$
- From part (a): $R_{kt} = R_{1t} + e_t$, where e_t is stationary
- Therefore $R_{kt} \sim I(1)$ (sum of $I(1)$ and $I(0)$)
- The linear combination: $R_{kt} - R_{1t} = e_t \sim I(0)$

Since both series are $I(1)$ and their difference is $I(0)$, they are cointegrated with **cointegrating coefficient = 1**.

(c) When $\Delta R_{1t} = \phi_1 \Delta R_{1,t-1} + u_t$ with $|\phi_1| < 1$:

- ΔR_{1t} is stationary (AR(1) with $|\phi_1| < 1$)
- But R_{1t} itself has a unit root—it follows an AR(2) with a unit autoregressive root:

$$R_{1t} - R_{1,t-1} = \phi_1(R_{1,t-1} - R_{1,t-2}) + u_t$$

$$R_{1t} = (1 + \phi_1)R_{1,t-1} - \phi_1 R_{1,t-2} + u_t$$

This is ARIMA(1,1,0), so $R_{1t} \sim I(1)$.

The i-period ahead forecast of ΔR_{1t}:

$$E_t(\Delta R_{1,t+i}) = \phi_1^i \Delta R_{1t}$$

The i-period ahead forecast of R_{1t}:

$$E_t(R_{1,t+i}) = R_{1t} + \sum_{j=1}^{i} E_t(\Delta R_{1,t+j}) = R_{1t} + \Delta R_{1t} \sum_{j=1}^{i} \phi_1^j$$

Thus:

$$R_{kt} = R_{1t} + c \cdot \Delta R_{1t} + e_t$$

where c is a constant depending on k and ϕ_1.

Since $R_{kt} - R_{1t} = c \cdot \Delta R_{1t} + e_t$ is $I(0)$ (both ΔR_{1t} and e_t are stationary), **yes, R_{kt} and R_{1t} are cointegrated with cointegrating coefficient = 1**.

(d) When $R_{1t} = 0.5 R_{1,t-1} + u_t$:

- R_{1t} is a stationary AR(1) process (since $|0.5| < 1$)
- Therefore $R_{1t} \sim I(0)$
- R_{kt} is a weighted average of expected future R_{1t} values plus e_t
- Since R_{1t} is stationary, R_{kt} is also stationary: $R_{kt} \sim I(0)$

No, R_{kt} and R_{1t} are NOT cointegrated.

Cointegration requires both series to be $I(1)$. Here both are $I(0)$, so the concept of cointegration does not apply.

Question 4. Consider the regression:

$$Y_t = 2.0 + 3.6X_{t+1} - 1.2X_t - 0.3X_{t-1} + u_t$$

(a) Rewrite this in the form $Y_t = \beta_0 + \beta X_t + \delta_{-1}\Delta X_{t+1} + \delta_0 \Delta X_t + u_t$ and identify β, δ_{-1}, and δ_0.

(b) Cointegration between Y_t and X_t requires that X_t is $I(1)$ and u_t is $I(0)$. For each case below, are Y_t and X_t cointegrated?

(i) $X_t \sim I(0)$, $u_t \sim I(0)$

(ii) $X_t \sim I(1)$, $u_t \sim I(1)$

(iii) $X_t \sim I(1)$, $u_t \sim I(0)$

Solution:

(a) Start with:

$$Y_t = 2.0 + 3.6X_{t+1} - 1.2X_t - 0.3X_{t-1} + u_t$$

Rewrite using $\Delta X_{t+1} = X_{t+1} - X_t$ and $\Delta X_t = X_t - X_{t-1}$:

$$\begin{aligned}
Y_t &= 2.0 + 3.6X_{t+1} - 1.2X_t - 0.3X_{t-1} + u_t \\
&= 2.0 + 3.6(X_t + \Delta X_{t+1}) - 1.2X_t - 0.3(X_t - \Delta X_t) + u_t \\
&= 2.0 + 3.6X_t + 3.6\Delta X_{t+1} - 1.2X_t - 0.3X_t + 0.3\Delta X_t + u_t \\
&= 2.0 + (3.6 - 1.2 - 0.3)X_t + 3.6\Delta X_{t+1} + 0.3\Delta X_t + u_t \\
&= 2.0 + 2.1X_t + 3.6\Delta X_{t+1} + 0.3\Delta X_t + u_t
\end{aligned}$$

But we want the form with $\Delta X_{t+1} = X_{t+1} - X_t$, written as leads/lags from X_t: Actually, using the convention δ_{-1} for the lead difference:

$$Y_t = \beta_0 + \beta X_t + \delta_{-1}(X_{t+1} - X_t) + \delta_0(X_t - X_{t-1}) + u_t$$

Expanding: $Y_t = \beta_0 + \beta X_t + \delta_{-1}X_{t+1} - \delta_{-1}X_t + \delta_0 X_t - \delta_0 X_{t-1} + u_t$
Collecting terms: $Y_t = \beta_0 + \delta_{-1}X_{t+1} + (\beta - \delta_{-1} + \delta_0)X_t - \delta_0 X_{t-1} + u_t$

Matching coefficients with $Y_t = 2.0 + 3.6X_{t+1} - 1.2X_t - 0.3X_{t-1} + u_t$:

$$\begin{aligned}
\delta_{-1} &= 3.6 \\
-\delta_0 &= -0.3 \Rightarrow \delta_0 = 0.3 \\
\beta - \delta_{-1} + \delta_0 &= -1.2 \\
\beta - 3.6 + 0.3 &= -1.2 \\
\beta &= -1.2 + 3.3 = 2.1
\end{aligned}$$

$$\boxed{\beta = 2.1, \quad \delta_{-1} = 3.6, \quad \delta_0 = 0.3}$$

Note: Some textbooks use different sign conventions. An alternative representation:

$$Y_t = 2.0 + 2.1X_t + 1.5(X_{t+1} - X_t) + 0.3(X_t - X_{t-1}) + u_t$$

would give $\delta_{-1} = 1.5$ if we factor differently.

(b) Cointegration requires: (i) both Y_t and X_t are $I(1)$, and (ii) there exists a linear combination that is $I(0)$.

From the regression: $u_t = Y_t - 2.0 - 2.1X_t - \delta_{-1}\Delta X_{t+1} - \delta_0 \Delta X_t$

If $X_t \sim I(1)$, then $\Delta X_t \sim I(0)$.

(i) $X_t \sim I(0)$, $u_t \sim I(0)$: **No.**
Cointegration requires $X_t \sim I(1)$. Since X_t is stationary, cointegration is not applicable.

(ii) $X_t \sim I(1)$, $u_t \sim I(1)$: **Cannot determine / No.**
Although $X_t \sim I(1)$, if $u_t \sim I(1)$ then there is no stationary linear combination, so Y_t and X_t are not cointegrated.

(iii) $X_t \sim I(1)$, $u_t \sim I(0)$: **Yes.**
Both conditions for cointegration are satisfied:

- $X_t \sim I(1)$ (and therefore $Y_t \sim I(1)$)
- The error $u_t = Y_t - \beta_0 - \beta X_t -$ stationary terms is $I(0)$

The cointegrating relationship is $Y_t - 2.1X_t \sim I(0)$ (up to stationary components).

Question 5. Consider the GARCH(1,1) model:

$$u_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.}(0, 1)$$

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2$$

where $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$.

(a) Show that the unconditional variance of u_t is $\text{var}(u_t) = \dfrac{\omega}{1 - \alpha - \beta}$.

(b) Derive $E(\sigma_t^2)$.

(c) What restriction on the parameters is needed for $\text{var}(u_t)$ to be positive and finite?

(d) For the ARCH(1) model ($\beta = 0$), derive the unconditional variance.

(e) What is the stationarity condition for ARCH(1)?

Solution:

(a) Using the law of iterated expectations:

$$\text{var}(u_t) = E(u_t^2) = E[E(u_t^2 | \mathcal{F}_{t-1})] = E(\sigma_t^2)$$

where $\mathcal{F}_{t-1}$ is the information set at $t - 1$.
Taking unconditional expectations of the GARCH equation:

$$E(\sigma_t^2) = \omega + \alpha E(u_{t-1}^2) + \beta E(\sigma_{t-1}^2)$$

By stationarity: $E(\sigma_t^2) = E(\sigma_{t-1}^2)$ and $E(u_t^2) = E(u_{t-1}^2) = E(\sigma_t^2)$.
Let $\sigma^2 = E(\sigma_t^2)$:

$$\sigma^2 = \omega + \alpha\sigma^2 + \beta\sigma^2 = \omega + (\alpha + \beta)\sigma^2$$

Solving:

$$\sigma^2(1 - \alpha - \beta) = \omega$$

$$\boxed{\text{var}(u_t) = E(\sigma_t^2) = \frac{\omega}{1 - \alpha - \beta}}$$

(b) From part (a), using the same derivation:

$$\boxed{E(\sigma_t^2) = \frac{\omega}{1 - \alpha - \beta}}$$

(c) For the unconditional variance to be positive and finite:

1. **Positive:** Since $\omega > 0$, we need $1 - \alpha - \beta > 0$

2. **Finite:** We need the denominator to be strictly positive

Therefore, the **stationarity condition** is:

$$\boxed{\alpha + \beta < 1}$$

Combined with the non-negativity constraints $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$, this ensures:

- The process is covariance stationary
- The unconditional variance exists and is positive
- Shocks to volatility decay over time (mean-reverting volatility)

(d) For ARCH(1) with $\beta = 0$:

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2$$

Taking expectations:

$$E(\sigma_t^2) = \omega + \alpha E(u_{t-1}^2) = \omega + \alpha E(\sigma_{t-1}^2)$$

By stationarity:

$$\sigma^2 = \omega + \alpha\sigma^2$$

$$\sigma^2(1 - \alpha) = \omega$$

$$\boxed{\text{var}(u_t) = \frac{\omega}{1 - \alpha}}$$

(e) For ARCH(1), the stationarity condition is:

$$\boxed{\alpha < 1}$$

This ensures:

- The unconditional variance $\omega/(1 - \alpha)$ is positive and finite
- The effect of past shocks on current volatility decays geometrically at rate α
- The process is weakly stationary (constant unconditional mean and variance)

Intuition: When $\alpha + \beta < 1$ in GARCH(1,1), volatility shocks are temporary—a large u_{t-1}^2 increases σ_t^2, but the effect diminishes over time. If $\alpha + \beta \geq 1$, shocks have permanent effects (integrated GARCH or explosive behavior).